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NOISY TRIGONOMETRIC MOMENTUM PROBLEM, SUPERRESOLUTION AND THE ANALYSIS OF THE STEP-FREQUENCY RADAR DATA.

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Step-frequency radar may be employed for measuring the structure of the one-dimensionally layered media. The main challenge then is the accurate and computationally efficient analysis of the data acquired by the radar. If multiple reflections may be ignored, the problem reduces to the classical inverse problem of reconstructing the function from a finite number of its Fourier coefficients perturbed by noise. The minimum norm solution given by the discrete Fourier transform does not allow resolving of the features closer than the Raleigh distance. The proposed non-linear algorithm employs the apparatus of Pad approximations and orthogonal polynomials to obtain the iteratively refined spectrum estimate. The published error analysis for the certain matrix algorithms can be adapted to evaluate the influence of the noise on our method of signal recovery. Computational costs are low enough to allow the reconstruction in real time on inexpensive hardware.

Statement of the problem

Analysis of the data from the step-frequency radar in important special cases reduces to the trigonometric momentum problem, or the problem of recovery of the spectrum distribution $\mathrm{f}(\nu)$ from the discrete set of moments

$$\left\{ s_n : s_n = \int_0^1 \cos(\pi n \nu)\,\mathrm{f}(\nu)\,d\nu,\ n = 0 \ldots N_\nu \right\}. \tag{1}$$

The discrete cosine transform $\mathrm{DCT}[s_n](k) = \alpha_k \sum_{n=0}^{N_\nu} s_n \cos\left((n+1/2)\pi k / N_\nu\right)$ provides a solution computable using fast (costing $O(N_\nu \log N_\nu)$ operations) linear algorithms. However, the resolution is limited by the Raleigh distance π / N_ν . Achieving greater resolution requires using more complex algorithms.

Stieltjes moment problem and the Lanczos algorithm

This section describes the widely known approach to estimating the unknown Stieltjes density $\mu(x)$ (see, for instance [4, § 7.8]). Its essence consists in constructing the sequence of polynomials $\pi_n(x)$ starting with $\pi_0 = 1$, orthonormal with respect to the scalar product

$$\langle \pi_k, \pi_l \rangle = \int \pi_k(x)\,\pi_l(x)\,d\,\mu(x), \tag{2}$$

using the three-term Lanczos recurrence

$$\begin{gathered} \Pi_k(x) = (x - \alpha_k)\,\pi_k(x) - \beta_{k-1}\,\pi_{k-1}(x), \quad \pi_{k+1}(x) = \Pi_k(x)/\beta_k, \text{ where} \\ \alpha_k = \langle (x-\alpha_k)\,\pi_k - \beta_{k-1}\,\pi_{k-1}, \pi_k \rangle, \quad \beta_k^2 = \langle \Pi_k, \Pi_k \rangle, \quad \beta_0 = 0. \end{gathered} \tag{3}$$

The knowledge of the first $N+1$ Stieltjes moments

$$\left\{ m_n : m_n = \int x^n\,d\,\mu(x),\ n = 0 \ldots N \right\} \tag{4}$$

suffices to compute the scalar products $\langle \pi_k, \pi_l \rangle$ for any $k + l \le N$, therefore enabling one to carry out the Lanczos iterations up to the order $k_{\max} = \lfloor N/2 \rfloor$. It can be shown then that a Stieltjes density function concentrated in the nodes u_i with the weights μ_i;

$$\mu(x) = \sum_{i=1}^{k_{\max}} \mu_i\,\theta(x - u_i), \qquad \theta(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}, \tag{5}$$

has the first $2k_{\max}$ moments coinciding with the set $\{m_n\}$ used to construct the set of polynomials $\{\pi_k : k = 1\ldots k_{\max}\}$, provided u_i are chosen equal to the eigenvalues of the tridiagonal matrix

$$T_k = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{k-1} \\ & & \beta_{k-1} & \alpha_k \end{pmatrix}, \qquad (6)$$

and μ_i proportional to the squared first components $\mu_i = \gamma s_{1i}^2$ of the corresponding eigenvectors $(s_{1i} \quad s_{2i} \quad \cdots \quad s_{ki})^T$. If the unknown Stieltjes density $\mu(x)$ is , in fact, discrete with K_S points of increase, the residual norm β_k becomes zero at iteration $k = K_S$.

From Stieltjes to trigonometric moment problem

Comparison of (1) and (4) suggests that the Stieltjes density $\mu(x)$ appear to parallel the unknown spectral density $\mathrm{F}(\nu)$:

$$s_n = \int_0^1 \cos(\pi n \nu)\,\mathrm{f}(\nu)d\nu = \int_0^1 \cos(\pi n \nu)d\,\mathrm{F}(\nu) = \int_{-1}^1 \mathrm{T}_n(x)d\,\mathrm{F}(x), \text{ where} \qquad (7)$$

$x = \cos(\pi\nu)$ and $\mathrm{T}_n(x) = \cos(n \arccos(x))$ is the Chebyshev polynomial of the 1st kind. As Chebyshev polynomials form a complete basis in the space of all polynomials, knowledge of the moments s_n allows to compute the moments

$$m_n = \int_{-1}^1 x^n d\,\mathrm{F}(x) \qquad (8)$$

and thus to carry out the Lanczos iterations (3). The eigenvalues and eigenvectors of the resulting tridiagonal matrix (6) enable one to form an approximation of the type (5) to the unknown density $\mathrm{F}(x)$.

Actual spectrum and the approximated spectrum

The approximation $\mu(x)$ to the unknown spectral density $\mathrm{F}(x)$ has been constructed, based on the knowledge of the moments (8) without making any a priory assumptions about it[1], but it is an approximation of the particular kind. If the measured signal has the discrete line spectrum of the form (5), the Lanczos process (3) will terminate with $\beta_k = 0$ at the iteration $k = k_{\max}$, recovering the unknown spectrum. Before the termination of the Lanczos process, or when the signal (1) has a continuous distribution $\mathrm{f}(\nu)$ (in this case the Lanczos process can be carried out to arbitrary number of iterations), connection between it and the approximation (5) is given by the Chebyshev inequalities for the density function. Namely, theorem 3.2.1 of [1, Part 2, § 3.2] implies[2] the inequality

$$\mu(u_i - 0) \le \mathrm{F}(x) \le \mu(u_i + 0), \qquad (9)$$

where $u_i,\ i = 1\ldots k_{\max}$ are the points of discontinuity of $\mu(x)$.

Influence of the noise

If the moments s_n are obtained by some measuring device, they may be perturbed by the unknown random noise: $s_n \to s_n + \varepsilon_n$. It is important to consider what influence this unknown perturbation of the input data will have on the described algorithm. Fortunately, there exist extensive literature concerning the numerical stability of Lanczos algorithm applied to the eigenvalue problem (See, for instance, [2], [3], [4,

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[1] Except that it is non-decreasing, thus ensuring that the norm induced by the scalar product (2) is nonnegative. In case $\mathrm{F}(x)$ has the points of decrease, we carry out the bi-Lanczos process.

[2] In the special case w is equal to a root of $B^{[M/M+1]}(z)$ which is not also a root of $B^{[M/M]}(z)$

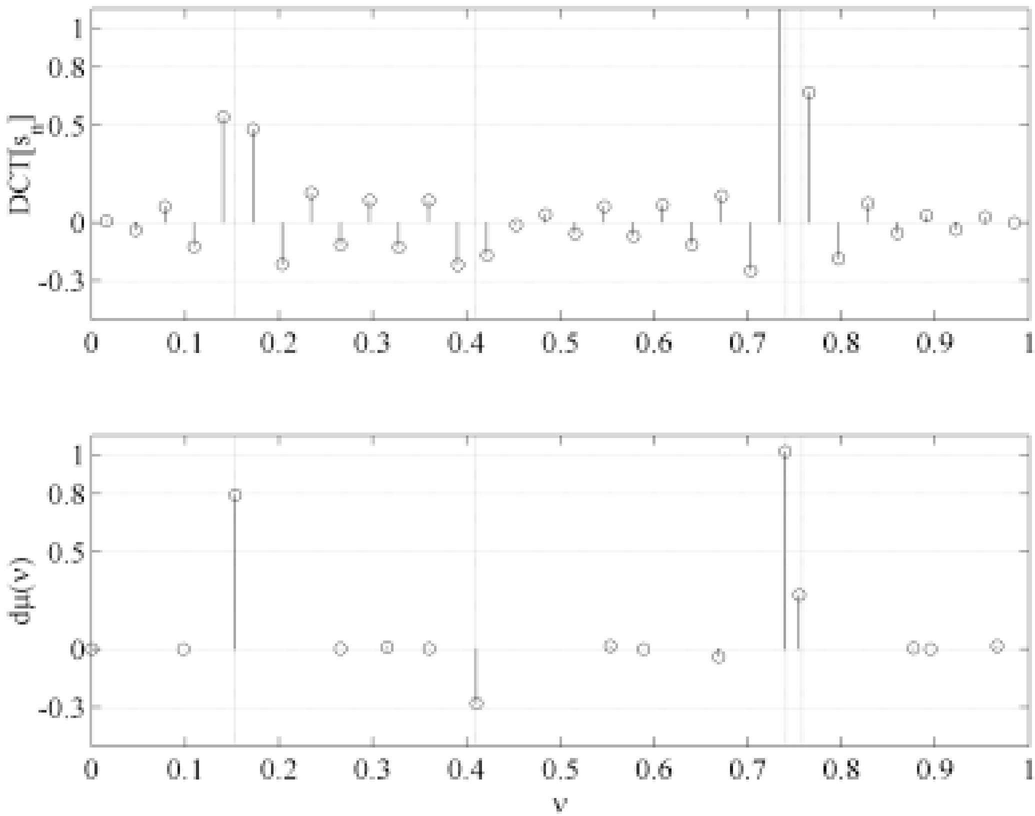
Ch. 13]). Those results concern the influence of the floating-point rounding errors. Rounding errors are analyzed as some unknown random perturbations of the Lanczos recurrence (3). Some of the results of the error analysis hold if rounding errors are replaced by the normally distributed random numbers with the standard deviation on the order of the machine ε (this idea was used in [5] to improve the numerical properties of the Lanczos algorithm). It is demonstrated that the eigenvalues recovered by the perturbed Lanczos iterations lie in tiny intervals around the true eigenvalues. According to [3], proven bounds on the size of those intervals considerably overestimate them, and are on the order of $\varepsilon^{1/4}$. Thus, we may expect that the described superresolution algorithm will recover the unknown frequencies present in the signal perturbed by noise with standard deviation ε with the error not exceeding $\varepsilon^{1/4}$. This is confirmed by the numerical experiments.

Numerical experiment

The algorithm was applied to the sequence

$$s_n = 0.8\cos(0.154\pi n) - 0.3\cos(0.41\pi n) + 1.0\cos(0.741\pi n) + 0.5\cos(0.758\pi n) + \varepsilon_n,$$

$n = 1\ldots32$, where ε_n is a pseudorandom sequence with zero mean and standard deviation 0.01. The graph shows the approximation $d\mu(\nu)$ of the spectral distribution $f(\nu)$, as well as the discrete cosine transform of the sequence s_n.



Dotted vertical lines show the actual location of the spectral lines. Two spectral lines closer than the Raleigh distance $1/32 = 0.0313$ are successfully resolved

Bibliography

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