STANDARD AND SINGULAR 1D AND 2D POSITIVE LINEAR SYSTEMS

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Abstract: Notions of the externally and internally positive standard and singular discrete-time and continuous-time linear systems are introduced. Necessary and sufficient conditions for the external and internal positivity of 1D and 2D linear systems are established. It is shown that the reachability and controllability of the internally positive 1D and 2D linear systems are not invariant under the state-feedbacks. By suitable choice of the state-feedbacks an unreachable internally positive linear systems can be made reachable and a controllable internally positive system can be made uncontrollable.

1. Introduction

The positive discrete-time and continuous-time linear systems have been considered in many papers [5,7,8,38,39]. The most popular models of two – dimensional (2D) linear systems are the models introduced by Roesser [41], Fornasini and Marchesini [9,10] and Kurek [32]. An overview of recent developments in reachability and controllability of 2D linear systems can be found in [28–31,25, 16]. The positive (non – negative) 2D Roesser type model has been introduced in [21] and its reachability and controllability has been considered in [21–23]. The reachability and controllability of weakly positive systems have been studied in [24, 26]. Some recent developments in 2D positive system theory have been given in [11, 42]. In this paper an overview of recent developments and new results in 1D and 2D positive linear systems will be presented. It is well – known [16, 28] that the reachability and controllability of standard linear 1D and 2D systems are invariant under the state – feedbacks. To the best author's knowledge the reachability and controllability of positive 1D and 2D linear systems with state – feedbacks have been not considered yet. In this paper it will be shown that the reachability and controllability of linear positive 1D and 2D systems are not invariant under the state – feedbacks.

Externally and internally positive standard 1D linear systems.

2.1. Discrete-time systems

Let $R^{n\times m}$ be the set of $n\times m$ matrices with entries from the field of real numbers R and $R^n := R^{n\times 1}$. The set of $n\times m$ matrices with real non-negative entries will be denoted by $R_+^{n\times m}$ and $R_+^n := R_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider the discrete-time linear system

(1a) $$Ex_{i+1} = Ax_i + Bu_i \quad , \quad i\in Z_+$$

(1b) $$y_i = Cx_i + Du_i$$

where $x_i \in R^n$, $u_i \in R^m$ and $y_i \in R^p$ are the state, input and output vectors and $E, A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}, D\in R^{p\times m}$.

The system (1) is called singular if $\det E = 0$. If $\det E \neq 0$ then premultiplying (1a) by E^{-1} we obtain the standard system

(2a) $$x_{i+1} = Ax_i + Bu_i \quad , \quad i\in Z_+$$

(2b) $$y_i = Cx_i + Du_i$$

For the singular system (1) it is assumed that

$$\det[Ez - A] \neq 0 \text{ for some } z\in \mathbf{C} \text{ (the field of complex numbers)}$$

Definition 1. The standard system (2) is called externally positive if for $x_0 = 0$ and every $u_i \in R_+^m$, $i\in Z_+$ we have $y_i \in R_+^p$ for $i\in Z_+$.

Theorem 1. [24] The standard system (2) is externally positive if and only if its impulse response matrix

(4) $$g_i = \begin{cases} CA^iB & for\ i>0 \\ D & for\ i=0 \end{cases}$$

is non-negative, $g_i \in R_+^{p\times m}$ for $i\in Z_+$

Definition 2. The standard system (2) is called internally positive if for every $x_0 \in R_+^n$ and all inputs $u_i \in R_+^m$, $i\in Z_+$ we have $x_i \in R_+^n$ and $y_i \in R_+^p$ for $i\in Z_+$.

Theorem 2. [24] The standard system (2) is internally positive if and only if

(5) $$A\in R_+^{n\times n}, B\in R_+^{n\times m}, C\in R_+^{p\times n}, D\in R_+^{p\times m}$$

The standard internally positive system (2) is always externally positive.

2.2. Continuous-time systems.

Consider the continuous-time linear system

(6a) $$E\dot{x} = Ax + Bu$$

(6b) $$y = Cx + Du$$

$\dot{x} = \frac{dx}{dt}$, $x = x(t) \in R^n$, $u = u(t) \in R^m$, $y = y(t) \in R^p$ are the state, input and output vectors, and $E, A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$.

The system (6) is called singular if $\det E = 0$. If $\det E \neq 0$ then premultiplying (6a) by E^{-1} we obtain the standard system

(7a) $$\dot{x} = Ax + Bu$$

(7b) $$y = Cx + Du$$

For the singular system (6) it is assumed that

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C}$$

Definition 3. The standard system (7) is called externally positive if for $x_0 = x(0) = 0$ and every $u(t) \in R_+^m$, $t \geq 0$ we have $y(t) \in R_+^p$, for $t \geq 0$.

Theorem 3. [24] The standard system (7) is externally positive if and only if its impulse response matrix

(9) $$g(t) = \begin{cases} Ce^{At}B & for\ t > 0 \\ D\delta(t) & for\ t = 0 \end{cases}$$

is non-negative, $g(t) \in R_+^{p \times m}$ for $t \geq 0$, where $\delta(t)$ is the Dirac impulse.

Definition 4. The standard system (7) is called internally positive if for every $x_0 \in R_+^n$ and all inputs $u(t) \in R_+^m$, $t \geq 0$ we have $x(t) \in R_+^n$ and $y(t) \in R_+^p$ for $t \geq 0$.

Theorem 4. [24] The standard system (7) is internally positive if and only if A is a Metzler matrix (all off-diagonal entries are non-negative) and $B \in R_+^{n \times m}$, $C \in R_+^{p \times n}$, $D \in R_+^{p \times m}$

The standard internally positive system (7) is always externally positive. The standard internally positive system (2) and (7) will be shortly called positive.

2.3. Reachability and controllability of positive 1D systems.

Definition 5. The positive system (2) is called h-step reachable if for every $x_f \in R_+^n$ (and $x_0 = 0$) there exists a input sequence $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = x_f$.

Definition 6. The positive system (2) is called reachable if for every $x_f \in R_+^n$ (and $x_0 = 0$) there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = x_f$.

Definition 7. The positive system (2) is called controllable if for every nonzero $x_f, x_0 \in R_+^n$ there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = x_f$.

Definition 8. The positive system (2) is called controllable to zero if for every $x_0 \in R_+^n$ there exists $h \in Z_+$ and $u_i \in R_+^m$, $i = 0,1,...,h-1$ such that $x_h = 0$.

Theorem 5. [7,24] The positive system (2) is n-step reachable if and only if:

rank $R_n = n$

there exists a nonsingular matrix $\overline{R}_n$ consisting of n columns of R_n such that $R_n^{-1} \in R_+^{n \times n}$ or equivalently R_n has n linearly independent columns each containing only one positive entry

where

(10) $$R_n := \left[B, AB, ..., A^{n-1}B\right] \in R_+^{n \times nm}$$

If the positive system (2) is reachable then it is always n-step reachable [7,8]. □

Theorem 6. [7,24] The positive system (2) is controllable if and only if:

the matrix R_n has n linearly independent columns each containing only one positive entry.

the spectral radius $\rho(A)$ of A is $\rho(A) < 1$ if the transfer from x_0 to x_f is allowed in an infinite number of steps and $\rho(A) = 0$ if the transfer from x_0 to x_f is required in a finite number of steps. □

Let us assume that for $m = 1$ the matrices A and B of (2) have the canonical form

(11) $$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \in R_+^{n \times n},\ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R_+^n$$

It is easy to see that for (11)

(12) $$rank\left[B, AB, ..., A^{n-1}B\right] = n$$

but the condition ii) of theorem 5 is not satisfied if at least one $a_i \neq 0$ for $i = 1,..., n-1$. In this case the positive system (2) with (11) is not n-step reachable.

Consider the system (2) with state-feedback

(13) $$u_i = v_i + Kx_i$$

where $K \in R^{1\times n}$ and v_i is the new input.

Substitution of (13) into (2) yields

(14) $$x_{i+1} = A_c x_i + Bv_i,\ i \in Z_+$$

where

(15) $$A_c = A + BK$$

For (11) and

(16) $$K = [a_0, a_1, ..., a_{n-1}]$$

the matrix (15) has the form

(17) $$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [a_0, a_1, ..., a_{n-1}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Using (17) we obtain

$$\left[B, A_c B, ..., A_c^{n-1}B\right] = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \hdashline 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Then the conditions of theorem 5 are satisfied and the closed-loop system is n-step reachable.

Therefore, the following theorem has been proved.

Theorem 7. Let the positive system (2) with (11) be not n-step reachable. Then the closed-loop system (14) with (17) is n-step reachable if the state-feedback gain matrix K has the form (16). □

Corollary 1. The n-step reachability of positive system (2) with (11) is not invariant under the state-feedback (13).

Remark 1. It is well-known [16] that if the pair (A, B) satisfies the condition (12) then it can be transformed by linear state transformation $\overline{x}_i = Px_i$, $\det P \neq 0$ to the canonical form (11)

$$\overline{A} = PAP^{-1},\ \overline{B} = PB$$

and

$$[\overline{B}, \overline{A}\,\overline{B}, ..., \overline{A}^{n-1}\overline{B}] = P[B, AB, ..., A^{n-1}B]$$

Note that the conditions of theorem 5 are satisfied if and only if P is a monomial matrix (in each row and column has only one positive entry and the remaining entries are zero).

Consider the single-input system (2) with matrices A,B in the canonical form (11). In a similar way as in the reachability case it can be shown that the condition i) of theorem 6 is not satisfied if at least one of the coefficients $a_i \neq 0$ for $i = 1,..., n-1$. In this case the positive system (2) with (11) is not controllable. The closed-loop system matrix (15) with (11) and state-feedback gain matrix (16) has the form (17). Note that the matrix (17) has all zero eigenvalues and its spectral radius $\rho(A_c) = 0$.

Therefore, the following theorem has been proved.

Theorem 8. Let the positive system (2) with (11) be not controllable. Then the closed-loop system (14) with (17) is controllable in a finite number of steps if the state-feedback gain matrix K has the form (16). □

The considerations can be extended with some modifications for continuous-time positive linear systems.

Externally and internally positive singular 1D linear systems.

3.1. Discrete-time systems.

Consider the singular discrete-time system (1) with $m = p = 1$ and

$$E = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{n\times n},\ A = \begin{bmatrix} 0 \ \vdots\ I_{n-1} \\ \hdashline a \end{bmatrix} \in R^{n\times n}, a = \begin{bmatrix} a_0 & a_1 \cdots a_{r-1} & -1 & 0 \cdots 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^n,\ C = \begin{bmatrix} b_0 & b_1 \cdots b_{n-1} \end{bmatrix} \in R^{1\times n},\ D = 0$$

If (3) holds then

$$[Ez - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i z^{-(i+1)}$$

where μ is the nilpotence index of (E,A) and Φ_i are the fundamental matrices satisfying the relation

$$E\Phi_i - A\Phi_{i-1} = \Phi_i E - \Phi_{i-1} A = \begin{cases} I \ (the \ \ identity \ \ matrix) \ \ for \ \ i = 0 \\ 0 \ (the \ \ \ zero \ \ matrix) \ \ for \ i \neq 0 \end{cases}$$

Theorem 9. If the matrices E,A,B,C have the canonical form (18) and

(19) $$a_i \geq 0, \ i = 0,1,...,r-1, \ \ b_j \geq 0, j = 0,1,...,n-1 \quad (n > r)$$

then

(20) $$\Phi_k B \in R_+^n \ \text{ for } \ k = -\mu, 1-\mu, ...$$

$$\Phi_i \in R_+^{n\times n} \ \text{ for } \ i \in Z_+$$

$$g_j \in R_+^{p\times m} \ \text{ for } \ j = 1-\mu, 2-\mu, ...$$

The proof is given in [15]

Theorem 10. The singular system (1) with (18) is externally and internally positive if (19) hold.
The proof follows from the relations (20)-(22).

3.2. Continuous-time systems.

Consider the singular continuous-time single-input single-output ($B = b,\ C = c,\ D = 0$) linear system (6). It is assumed that $\det E = 0$ and

(23 $$\det[Es - A] \neq 0 \ \text{ for some } \ s \in \mathbf{C}$$

If (23) holds then

(24) $$[Es - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i s^{-(i+1)}$$

where μ is the nilpotence index of (E, A) and Φ_i are the fundamental matrices defined by

$$E\Phi_i - A\Phi_{i-1} = \Phi_i E - \Phi_{i-1} A = \begin{cases} I \ (the \ \ identity \ \ matrix) \ \ for \ \ i = 0 \\ 0 \ (the \ \ \ zero \ \ matrix) \ \ for \ i \neq 0 \end{cases}$$

and $\Phi_i = 0$ for $i < -\mu$.

Using (25) it is easy to show that [37]

(26a) $$\Phi_0 A \Phi_i = \begin{cases} \Phi_{i+1} \ \ for \ \ i \geq 0 \\ 0 \ for \ i < 0 \end{cases}$$

and

(26b) $$-\Phi_{-1} E \Phi_i = \begin{cases} 0 \ \ for \ \ i \geq 0 \\ \Phi_{i-1} \ \ for \ i < 0 \end{cases}$$

The solution $x(t)$ of the equation (6a) with initial conditions x_0 is given by [26]

(27) $$x(t) = e^{\Phi_0 At}\Phi_0 Ex_0 + \int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 bu(\tau)d\tau + \sum_{j=1}^{\mu}\Phi_{-j}\left(bu^{(j-1)}(t) + Ex_0\delta^{(j-1)}(t)\right)$$

and

(28) $$y(t) = cx(t) = ce^{\Phi_0 At}\Phi_0 Ex_0 + \int_0^t ce^{\Phi_0 A(t-\tau)}\Phi_0 bu(\tau)d\tau + \sum_{j=1}^{\mu} c\Phi_{-j}\left(bu^{(j-1)}(t) + Ex_0\delta^{(j-1)}(t)\right)$$

where $u^{(j)} = \dfrac{d^j u}{dt^j} \quad j = 1,...,\mu - 1$.

Substituting $x_0 = 0$ and $u(t) = \delta(t)$ into (28) we obtain the impulse response $g(t)$ of the system

(29) $$g(t) = \begin{cases} ce^{\Phi_0 At}\Phi_0 b \quad for \ \ t > 0 \\ ce^{\Phi_0 At}\Phi_0 b + \sum_{j=1}^{\mu} c\Phi_{-j} b\delta^{(j-1)}(t) \quad for \quad t = 0 \end{cases}$$

The transfer function of the system is given by

(30a)
$$T(s) = c[Es - A]^{-1} b$$

The substitution of (24) into (30a) yields

(30b)
$$T(s) = \sum_{i=0}^{\infty} c\Phi_i b s^{-(i+1)} + \sum_{j=1}^{\mu} c\Phi_{-j} b s^{j-1}$$

From (26a) for $i \ge 0$ we have $\Phi_1 = \Phi_0 A \Phi_0$, $\Phi_2 = \Phi_0 A \Phi_1 = (\Phi_0 A)^2 \Phi_0$ and

$$\Phi_i = (\Phi_0 A)^i \Phi_0 \quad \text{for} \quad i \ge 0$$

Substituting (31) into (30b) we obtain

(32)
$$T(s) = c\left[\sum_{i=0}^{\infty} (\Phi_0 A)^i s^{-(i+1)}\right]\Phi_0 b + \sum_{j=1}^{\mu} c\Phi_{-j} b s^{j-1}$$

Applying to (32) the Laplace inverse transform we obtain (29).

Using the impulse response (29) the formula (28) for $t > 0$ can be rewritten in the form

(33)
$$y(t) = ce^{\Phi_0 A t}\Phi_0 E x_0 + \int_0^t g(t-\tau)u(\tau)d\tau + \sum_{j=1}^{\mu} c\Phi_{-j} b u^{(j-1)}(t) \quad \text{for } t > 0$$

Definition 9. The singular system (6) is called externally positive if for $x_0 = 0$ and any nonnegative input $u(t) \ge 0$ with $u^{(j)}(t) \ge 0$ for $j = 1, \ldots, \mu - 1$ for $t \in R_+$ the output $y(t)$ is also nonnegative, $y(t) \ge 0$ for $t > 0$.

Theorem 11. The singular system (6) with $B = b$, $C = c$, $D = 0$ is externally positive if and only if its impulse response $g(t)$ is nonnegative

(34)
$$g(t) \in R_+ \quad \text{for} \quad t \in R_+$$

Proof. The necessity follows immediately from the definition of impulse response and the definition 9. To prove the sufficiency we assume that (34) holds. Then from (33) for $x_0 = 0$ and $u^{(j)}(t) \ge 0$ for $j = 0, 1, \ldots, \mu - 1$ for $t \in R_+$ we obtain $y(t) \ge 0$ for $t > 0$. □

Example 1. Consider the system (6) with the matrices

(35)
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -1 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}$$

and $a \ge 0, b_i \ge 0, i = 0, 1, 2$.

In this case using (24) and (31) we obtain

(36a)
$$[Es - A]^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -a & 1 & 0 \end{bmatrix}^{-1} = \frac{1}{s-a}\begin{bmatrix} 1 & 0 & 1 \\ a & 0 & s \\ as & a-s & s^2 \end{bmatrix} = \Phi_{-2}s + \Phi_{-1} + \Phi_0 s^{-1} + \Phi_1 s^{-2} + \cdots$$

where

$$\Phi_{-2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Phi_{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ a & -1 & a \end{bmatrix}, \Phi_0 = \begin{bmatrix} 1 & 0 & 1 \\ a & 0 & a \\ a^2 & 0 & a^2 \end{bmatrix},$$

(36b)
$$\Phi_i = (\Phi_0 A)^i \Phi_0 \quad for \quad i \ge 1, \; \Phi_0 A = \begin{bmatrix} a & 0 & 0 \\ a^2 & 0 & 0 \\ a^3 & 0 & 0 \end{bmatrix},$$

To find the impulse response of the system we calculate

(37)
$$e^{\Phi_0 A t} = \begin{bmatrix} e^{at} & 0 & 0 \\ a(e^{at} - 1) & 1 & 0 \\ a^2(e^{at} - 1) & 0 & 1 \end{bmatrix}$$

and

(38a)
$$ce^{\Phi_0 A t}\Phi_0 b = (b_0 + b_1 a + b_2 a^2)e^{at} \quad \text{for} \quad t \ge 0$$

(38b) $$\sum_{j=1}^{\mu} c\Phi_j b\delta^{(j-1)}(t) = c\Phi_{-1}b\delta(t) + c\Phi_{-2}b\delta^{(1)}(t) = (b_1 + b_2)\delta(t) + b_2\delta^{(1)}(t) \quad \text{for } t = 0$$

From (38) it follows that if $a \geq 0$ and $b_i \geq 0$ for $i = 0,1,2$ then the impulse response $g(t)$ of the system satisfies the condition (34). Thus, the system is externally positive.

Consider the singular system (6) with (18).

The initial conditions x_0 of the system (6) with (18) are called admissible for $u = 0$ if $ax_0 = a_0x_{10} + a_1x_{20} + \cdots + a_{r-1}x_{r0} - x_{r+1,0} = 0$, where $x_0 = [x_{10} \;\; x_{20} \cdots x_{n0}]^T$ (T denotes the transpose)

It is easy to show that if $ax_0 = 0$ then

$$\sum_{j=1}^{\mu} \Phi_j Ex_0 = \sum_{j=2}^{\mu} \Phi_{-j} Ax_0 = 0$$

and the solution of (6a) with admissible initial conditions has the form (33) for $t > 0$.

Theorem 12. If

$$a_i \geq 0 \quad \text{for} \quad i = 0,1,\ldots,r-1 \quad \text{and} \quad b_j \geq 0 \quad \text{for} \quad j = 0,1,\ldots,n-1$$

for the matrices (18) then

(40) $$\Phi_i b \in R_+^n \quad \text{for} \quad i \geq -\mu$$

and

(41) $$\Phi_i \in R_+^{n\times n} \quad \text{for} \quad i \geq 0$$

Proof. If E,A and b have the canonical form (18) then it is easy to show that

(42a) $$[Es - A]_{ad} b = \frac{1}{d(s)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} = H_{n-1}bs^{n-1} + \cdots + H_1bs + H_0b$$

where

(42b) $$H_{n-1}b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ldots, H_1b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; H_0b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Using the well-known equality $[Es - A]_{ad} = (\det[Es - A])[Es - A]^{-1}$ we may write

(43)
$$H_{n-1}s^{n-1} + \cdots + H_1s + H_0 = \left(s^r - a_{r-1}s^{r-1} - \cdots - a_1s - a_0\right)\left(\Phi_{-\mu}s^{\mu-1} + \cdots + \Phi_{-1} + \Phi_0s^{-1} + \Phi_1s^{-2} + \cdots\right)$$

The comparison of the coefficients at the same powers of s^k for $n-1, n-2, \ldots, 0$ of (43) yields

$$\Phi_{-\mu} = H_{n-1}, \Phi_{1-\mu} = H_{n-2} + a_{r-1}H_{n-1}, \; \Phi_{2-\mu} = H_{n-3} + a_{r-1}H_{n-2} + \left(a_{r-1}^2 + a_{r-2}\right)H_{n-1}$$

and

(44) $$\begin{bmatrix} \Phi_{-\mu} \\ \Phi_{1-\mu} \\ \Phi_{2-\mu} \\ \cdots\cdots \\ \Phi_{r-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q_1 & 1 & 0 & \cdots & 0 & 0 \\ q_2 & q_1 & 1 & \cdots & 0 & 0 \\ \hdashline q_{n-1} & q_{n-2} & q_{n-3} & \cdots & q_1 & 1 \end{bmatrix} \begin{bmatrix} H_{n-1} \\ H_{n-2} \\ H_{n-3} \\ \cdots\cdots \\ H_0 \end{bmatrix}$$

where $r = n - \mu$ and

$$q_k := \sum_{i=1}^{k} a_{r-i}q_{k-i} \quad \text{for} \quad k = 1,2 \quad (q_0 := 1)$$

Comparing the coefficients of (43) at $s^{-1}, s^{-2}, \ldots$ we obtain

$$\Phi_r = a_{r-1}\Phi_{r-1} + a_{r-2}\Phi_{r-2} + \cdots + a_0\Phi_0$$
$$\Phi_{r+1} = a_{r-1}\Phi_r + a_{r-2}\Phi_{r-1} + \cdots + a_0\Phi_1$$

and

(46) $$\Phi_{r+k}b = \sum_{j=1}^{r} a_{r-j}\Phi_{r+k-j}b \quad \text{for } k = 0,1,...$$

From (42b),(44),(45) and (46) it follows that $\Phi_k b \in R_+^n$ for $k = -\mu, 1-\mu, ...$ since $H_j b \in R_+^n$ for $j = n-1, n-2, ..., 0,\ q_k \geq 0$ for $k = 1,2,...$ and $a_i \geq 0$ for $i = 0,1,...,r-1$.

Using the equality $[Es - A][Es - A]_{ad} = I_n \det[Es - A]$ we may write

(47) $$[Es - A]\left[H_{n-1}s^{n-1} + \cdots + H_1 s + H_0\right] = I_n\left(s^r - a_{r-1}s^{r-1} - \cdots - a_1 s - a_0\right)$$

The comparison of the coefficients at the same powers of s of (47) yields

$$AH_0 = a_0 I_n, AH_1 = EH_0 + a_1 I_n, ..., AH_{r-1} = EH_{r-2} + a_{r-1}I_n, AH_r = EH_{r-1} - I,$$
$$AH_{r+1} = EH_r, ..., AH_{n-1} = EH_{n-2}, EH_{n-1} = 0$$

The matrix

(48) $$H_0 = \left[\begin{array}{c:c} -a^{(0)} & \\ \hdashline a_0 I_{n-1} & e_1 \end{array}\right],\ a^{(0)} := [a_1 \ \ a_2 \cdots a_{r-1} - 1 \ \ 0 \cdots 0]$$

satisfies the equality $AH_0 = a_0 I_n$, where e_i is the ith column of I_n.

Using (18) it is easy to show that

(49)

$$[Es - A]_{ad} = \left[\begin{array}{cccccc} s^{r-1} - a_{r-1}s^{r-2} - \cdots - a_1 & s^{r-2} - a_{r-1}s^{r-3} - \cdots - a_2 & \cdots & 0 & 0 & 1 \\ a_0 & s(s^{r-2} - a_{r-1}s^{r-3} - \cdots - a_2) & \cdots & 0 & 0 & s \\ a_0 s & a_1 s + a_0 & \cdots & 0 & 0 & s^2 \\ a_0 s^2 & s(a_1 s + a_0) & \cdots & 0 & 0 & s^3 \\ \hdashline a_0 s^{n-3} & s^{n-4}(a_1 s + a_0) & \cdots & -p(s) & 0 & s^{n-2} \\ a_0 s^{n-2} & s^{n-3}(a_1 s + a_0) & \cdots & -sp(s) & -p(s) & s^{n-1} \end{array}\right] =$$
$$= H_{n-1}s^{n-1} + \cdots + H_1 s + H_0$$

where $p(s) = \det[Es - A]$.

Comparison of the coefficients at the same powers of s^k for $k = 0,1,...,n-1$ of (49) yields

$$H_i = \left[\begin{array}{c:c} -a^{(1)} & \\ \vdots & \\ -a^{(i+1)} & e_{i+1} \\ \overline{a}^{(1)} & \\ \vdots & \\ \overline{a}^{(q-i)} & \end{array}\right] \quad \text{for } i = 1,...,r-1$$

$$a^{(i)} = \left[\overbrace{0 \cdots 0}^{i-1} \ \ a_{i+1} \cdots a_{r-1} - 1 \ \ 0 \cdots 0\right]$$

(50) $$\overline{a}^{(j)} := \left[\overbrace{0 \cdots 0}^{j-1} \ \ a_0 \cdots a_i \ \ 0 \cdots 0\right], \quad j = 1,...,q-i$$

$$H_i = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \\ & \hat{a}^{(1)} & & \\ & \vdots & & \\ & \hat{a}^{(n-i-1)} & & \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} i+1 \quad \text{for } i = r,\dots,n-2$$

$$H_{n-1} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \hdashline 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

From (44), (45) and (50) we have

$$\Phi_0 = q_\mu H_{n-1} + \cdots + q_1 H_r + H_{r-1} = \left[\begin{array}{c:c:c} & & 0 \\ I_r & & \vdots \\ & & 0 \\ \hdashline & 0 & q_0 \\ W & & q_1 \\ & & \vdots \\ & & q_{n-r} \end{array}\right] \in R_+^{n\times n}$$

and

(51) $$\Phi_0 A = q_\mu H_{n-1} A + \cdots + q_1 H_r A + H_{r-1} A = \left(q_\mu H_{n-2} + \cdots + q_1 H_{r-3} + H_{r-2}\right)E =$$

$$= \left[\begin{array}{c:c:c:c} 0 & & & 0 \\ \vdots & I_{r-1} & & \vdots \\ 0 & & 0 & 0 \\ \hdashline & & & q_0 \\ & W & & \vdots \\ & & & q_{n-r} \\ \hdashline 0 & \cdots & 0 & & 0 \end{array}\right] \in R_+^{n\times n}$$

where $W = [w_{ij}] \in R_+^{(n-r)\times r}$, $w_{ij} = \sum_{l=1}^{j} a_{j-l} q_{i-l}$

From (31) and (51) it follows

$$\Phi_i = (\Phi_0 A)^i \Phi_0 \in R_+^{n\times n} \quad \text{for } i = 1,2,\dots \quad \square$$

Theorem 13. The singular system (6) with (18) is externally positive if the conditions (39) are satisfied.

Proof. By theorem 11 the system (6) with (18) is externally positive if and only if (34) holds. From theorem 12 it follows that if the conditions (39) are satisfied then $\Phi_0 A$ is the Metzler matrix. Taking into account that [24]

$$e^{\Phi_0 A t} \in R_+^{n\times n} \quad \text{for } t \ge 0$$

we obtain

(53) $$c e^{\Phi_0 A t} b \in R_+$$

and

$$y(t) \in R_+ \quad \text{for } t \ge 0$$

Thus if the conditions (39) are satisfied then (34) holds and the singular system (6) with (18) is externally positive. □

Definition 10. The singular system (6) is called internally positive if for every admissible $x_0 \in R_+^n$ and any nonnegative input $u(t) \ge 0$ with $u^{(j)}(t) \ge 0$, $j = 1,\dots,\mu-1$ for $t \in R_+$, the state vector $x(t) \in R_+^n$ and $y(t) \in R_+^p$ for $t > 0$.

From comparison of the definitions 9 and 10 it follows that every singular system (6) internally positive is always externally positive.

Theorem 14. The singular system (6) with (18) is internally positive if the conditions (39) are satisfied.

Proof. If the conditions (39) are satisfied then using (27),(28) and (52) we obtain

$$x(t)=e^{\Phi_0 At}\Phi_0 Ex_0+\int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 bu(\tau)d\tau+\sum_{j=1}^{\mu}\Phi_{-j}bu^{(j-1)}(t)\in R_+^n \text{ for } t\in R_+$$

and

$$y(t)=ce^{\Phi_0 At}\Phi_0 Ex_0+\int_0^t ce^{\Phi_0 A(t-\tau)}\Phi_0 bu(\tau)d\tau+\sum_{j=1}^{\mu}c\Phi_{-j}bu^{(j-1)}(t)\in R_+^n \text{ for } t\in R_+$$

since $e^{\Phi_0 At}\in R_+^{n\times n}, \Phi_0 E\in R_+^{n\times n}, \Phi_0 b\in R_+^n, \Phi_{-j}b\in R_+^n$ for $j=1,\ldots,\mu$.

Therefore, by definition 10 the system (6) with (18) is internally positive. □

Example 2. (Continuation of Example 1)

It will be shown that the system (6) with (35) is also internally positive if

(54) $$a\geq 0 \text{ and } b_2>0,\ b_1\geq 0, b_2\geq 0$$

If the conditions (54) are satisfied then using (27), (36) and (37) we obtain

$$x(t)=e^{\Phi_0 At}\Phi_0 Ex_0+\int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 bu(\tau)+\sum_{j=1}^{\mu}\Phi_{-j}(bu^{(j-1)}(t)+Ex_0\delta^{(j-1)}(t)=$$

$$=\begin{bmatrix}e^{at}\\ ae^{at}\\ a^2e^{at}\end{bmatrix}x_{10}+\int_0^t\begin{bmatrix}e^{a(t-\tau)}\\ ae^{a(t-\tau)}\\ a^2e^{a(t-\tau)}\end{bmatrix}u(\tau)d\tau+\begin{bmatrix}0\\1\\a\end{bmatrix}u(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}\dot{u}(t)+\begin{bmatrix}0\\0\\ax_{10}-x_{20}\end{bmatrix}\delta(t)\in R_+^3$$

since for admissible initial conditions $ax_{10}-x_{20}=0$.

Similarly using (28),(36), (37) and (38) we obtain

$$y(t)=ce^{\Phi_0 At}\Phi_0 Ex_0+\int_0^t ce^{\Phi_0 A(t-\tau)}\Phi_0 bu(\tau)+\sum_{j=1}^{\mu}c\Phi_{-j}(bu^{(j-1)}(t)+Ex_0\delta^{(j-1)}(t))=$$

$$=\left(b_0+b_1a+b_2a^2\right)e^{at}x_{10}+\int_0^t\left(b_0+b_1a+b_2a^2\right)e^{a(t-\tau)}u(\tau)d\tau+\left(b_1+b_2a\right)u(t)+b_2\dot{u}(t)\in R^+$$

for $t\geq 0$.

Consider the system (6) with

(55) $$E=\begin{bmatrix}I_n & 0\\ 0 & 0\end{bmatrix}\in R^{n\times n},\ A=\begin{bmatrix}A_1\\A_2\end{bmatrix},\quad \begin{matrix}A_1\in R^{(n-1)\times n}\\ A_2=[a_{n1}\ \ a_{n-2}\cdots a_{nn}]\end{matrix}$$

From (6a) and (55) for $t=0$ we have

(56) $$0=A_2x_0+B_2u(0)$$

The equation (56) determines the set of admissible initial conditions x_0 for a given input $u(t)$. Note that the assumption (23) implies that A_2 is not zero row and the singularity of the system implies that at least $a_{nn}=0$. From (56) for $u(0)=0$ it follows that the equation $A_2x_0=0$ for $x_0\in R_+^n,\ x_0\neq 0$ may be satisfied if A_2 contains at least one negative entry. Therefore, we have the following important corollaries.

Corollary 2. The singular system (6) with (55) is not internally positive if A is a Metzler matrix.

Corollary 3. The singular weakly positive [26] system (6) with (55) is not internally positive.

Externally and internally positive 2D linear systems.

4.1. Externally and internally positive Roesser model.

Consider the 2D Roesser model [41]

(57a) $$\begin{bmatrix}x_{i+1,j}^h\\ x_{i,j+1}^v\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_{ij}^h\\ x_{ij}^v\end{bmatrix}+\begin{bmatrix}B_1\\B_2\end{bmatrix}u_{ij}$$

(57b) $$y_{ij}=\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_{ij}^h\\ x_{ij}^v\end{bmatrix}+Du_{ij},\ i,j\in Z_+:=\{0,1,\ldots\}$$

where $x_{ij}^h\in R^{n_1}$ and $x_{ij}^v\in R^{n_2}$ are the horizontal and vertical state vectors at the point (i,j), respectively, $u_{ij}\in R^m$ is the input vector, $y_{ij}\in R^p$ is the output vector and $A_{kl}\in R^{n_k\times n_l}, B_k\in R^{n_k\times m}$, $C_k\in R^{p\times n_k}$, $k,l=1,2$, $D\in R^{p\times m}$.

Definition 11. The Roesser model (57) is called externally positive if for zero boundary conditions $x_{0j}^h = 0, j \in Z_+$, $x_{i0}^v = 0, i \in Z_+$ and all inputs $u_{ij} \in R_+^m$, $i, j \in Z_+$ we have $y_{ij} \in R_+^p$ for $i, j \in Z_+$

Theorem 15. [14] The Roesser model (57) is externally positive if and only if its impulse response matrix $g_{ij} \in R_+^{p\times m}$ for $i, j \in Z_+$.

Definition 12. The Roesser model (57) is called internally positive (shortly positive) if for all boundary conditions

(58) $$x_{0j}^h \in R_+^{n_1}, j \in Z_+ \quad \text{and} \quad x_{i0}^v \in R_+^{n_2}, i \in Z_+$$

and all $u_{ij} \in R_+^m$, $i, j \in Z_+$ we have $x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \in R_+^n$, $n = n_1 + n_2$ and $y_{ij} \in R_+^p$ for all $i, j \in Z_+$.

Theorem 16. [21] The Roesser model (57) is internally positive if and only if

(59) $$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R_+^{n\times n}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in R_+^{n\times m},\ C = [C_1 \ C_2] \in R_+^{p\times n}, D \in R_+^{p\times m}$$

The transition matrix T_{ij} for (57) is defined as follows

(60) $$T_{ij} = \begin{cases} I_n \quad for \quad i = j = 0 \\ T_{10}T_{i-1,j} + T_{01}T_{i,j-1}\ for \quad i, j \ge 0\,(i + j \ne 0) \\ T_{ij} = 0 \quad for \quad i < 0 \ \ or/and \ \ j < 0 \end{cases}$$

where

$$T_{10} := \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, T_{01} := \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

From (60) it follows that the transition matrix T_{ij} of the internally positive model (57) is a positive matrix, $T_{ij} \in R_+^{n\times n}$ for all $i, j \in Z_+$.

Definition 13. The internally positive Roesser model (57) is called reachable for zero boundary conditions (58) (ZBC) at the point $(h,k), (h,k \in Z_+, h,k > 0)$, if for every $x_f \in R_+^n$ there exists a sequence of inputs $u_{ij} \in R_+^m$ for $(i, j) \in D_{hk}$ such that $x_{hk} = x_f$, where

(61) $$D_{hk} := \begin{Bmatrix} (i, j) \in Z_+ \times Z_+ : 0 \le i \le h, 0 \le j \le k; \\ i + j \ne h + k \end{Bmatrix}$$

Definition 14. The internally positive Roesser model (57) is called controllable to zero (shortly controllable) at the point $(h,k), (h,k \in Z_+, h,k > 0)$ if for any nonzero boundary conditions

(62) $$x_{0j}^h \in R_+^{n_1},\ 0 \le j \le k \ \text{ and } \ x_{i0}^v \in R_+^{n_2},\ 0 \le i \le h$$

there exists a sequence of inputs $u_{ij} \in R_+^m$ for $(i, j) \in D_{hk}$ such that $x_{hk} = 0$.

Theorem 17. [21] The internally positive Roesser model (57) is reachable for ZBC at the point (h,k) if and only if there exists a monomial matrix R_n consisting of *n* linearly independent columns of the reachability matrix

(63) $$R_{hk} := [M_{hk}, M_{h-1,k}, M_{h,k-1}, \ldots, M_{01}, M_{10}]$$

where

(64) $$M_{ij} := T_{i-1,j}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{i,j-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

and T_{ij} is defined by (60).

Theorem 18. [21] The internally positive Roesser model (57) is controllable if and only if the matrix A is nilpotent matrix, i.e.

(65) $$\det\begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2 - A_{22} \end{bmatrix} = z_1^{n_1} z_2^{n_2}$$

To simplify the notation we assume that $m = 1$ (the single-input systems) and the matrices A and B of the internally positive model (57) have the canonical form [25]

$$A_{11}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \hdashline 0&0&0&\cdots&1\\a_1&a_2&a_3&\cdots&a_{n_1}\end{bmatrix},\ A_{12}=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\\ \hdashline 0&0&\cdots&0\\1&0&\cdots&0\end{bmatrix},\ B_1=\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}$$

(66)

$$A_{21}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n_1}\\a_{21}&a_{22}&\cdots&a_{2n_1}\\ \hdashline a_{n_2 1}&a_{n_2 2}&\cdots&a_{n_2 n_1}\end{bmatrix},\ A_{22}=\begin{bmatrix}b_1&1&0&\cdots&0&0\\b_2&0&1&\cdots&0&0\\ \hdashline b_{n_2-1}&0&0&\cdots&0&1\\b_{n_2}&0&0&\cdots&0&0\end{bmatrix},\ B_2=\begin{bmatrix}b_1\\b_2\\ \vdots\\b_{n_2-1}\\b_{n_2}\end{bmatrix}$$

where $a_l \geq 0, a_{kl} \geq 0, b_k \geq 0$ for $k=1,...,n_2$, $l=1,...,n_1$.

Consider the Roesser model (57) with the state-feedback

(67) $$u_{ij}=v_{ij}+K\begin{bmatrix}x_{ij}^h\\x_{ij}^v\end{bmatrix},\ i,j\in Z_+$$

where $K=[K_1,K_2]$, $K_1\in R^{1\times n_1}$, $K_2\in R^{1\times n_2}$ and $v_{ij}\in R^m$ is a new input vector. Substitution of (67) into (57a) yields

(68) $$\begin{bmatrix}x_{i+1,j}^h\\x_{i,j+1}^v\end{bmatrix}=A_c\begin{bmatrix}x_{ij}^h\\x_{ij}^v\end{bmatrix}+Bv_{ij}$$

where

(69) $$A_c=A+BK=\begin{bmatrix}A_{11}+B_1K_1, & A_{12}+B_1K_2\\A_{21}+B_2K_1, & A_{22}+B_2K_2\end{bmatrix}$$

The standard closed-loop system (68) is reachable (controllable) if and only if the standard 2D Roesser model (57) is reachable (controllable). It is easy to show that if at least one of $a_l \neq 0, l=1,...,n_1$ or $b_k \neq 0, k=1,...,n_2$ then the condition of theorem 17 is not satisfied and the positive model (57) is not reachable at the point (n_1,n_2). Let the positive system (57) with (66) be unreachable at the point (n_1,n_2). It will be shown that there exists a state-feedback gain matrix K such the closed-loop system (68) is reachable at the point (n_1,n_2).

Let

(70) $$K=[-a_1,-a_2,...,-a_{n_1},-1,0,...,0]$$

For (66) and (70) the matrix (69) has the form

(71a) $$A_c=A+BK=\begin{bmatrix}\overline{A}_{11}, \overline{A}_{12}\\ \overline{A}_{21}, \overline{A}_{22}\end{bmatrix}$$

where

$$\overline{A}_{11}=A_{11}+B_1K_1=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \hdashline 0&0&0&\cdots&1\\a_0&a_1&a_2&\cdots&a_{n_1}\end{bmatrix}+\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}[-a_0,-a_1,...,-a_{n_1}]=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \hdashline 0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix}$$

$$\overline{A}_{12}=A_{12}+B_1K_2=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\\ \hdashline 0&0&\cdots&0\\1&0&\cdots&0\end{bmatrix}+\begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}[-1\quad 0\ ...\ 0]=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\\ \hdashline 0&0&\cdots&0\\0&0&\cdots&0\end{bmatrix}$$

(71b)

$$\overline{A}_{21}=A_{21}+B_2K_1=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n_1}\\a_{21}&a_{22}&\cdots&a_{2n_1}\\ \hdashline a_{n_2 1}&a_{n_2 2}&\cdots&a_{n_2 n_1}\end{bmatrix}+\begin{bmatrix}b_1\\b_2\\ \vdots\\b_{n_2}\end{bmatrix}[-a_0,-a_1,...,-a_{n_1}]=\begin{bmatrix}\overline{a}_{11}&\overline{a}_{12}&\cdots&\overline{a}_{1n_1}\\ \overline{a}_{21}&\overline{a}_{22}&\cdots&\overline{a}_{2n_1}\\ \hdashline \overline{a}_{n_2 1}&\overline{a}_{n_2 2}&\cdots&\overline{a}_{n_2 n_1}\end{bmatrix}$$

$$\overline{A}_{22} = A_{22} + B_2 K_2 = \begin{bmatrix} b_1 & 1 & 0 & \cdots & 0 & 0 \\ b_2 & 0 & 1 & \cdots & 0 & 0 \\ \hdashline b_{n_2-1} & 0 & 0 & \cdots & 0 & 1 \\ b_{n_2} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_2} \end{bmatrix} [-1 \ 0 \cdots 0] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

If the assumptions of the canonical form are satisfied then it can be shown that $\overline{a}_{kl} \geq 0$ for $k = 1,...,n_2$, $l = 1,...,n_1$. Now we shall show that the closed-loop system with (71b) and $b_1 = b_2 = \cdots = b_{n_2-1} = 0, b_{n_2} \neq 0$ is reachable at the point (n_1, n_2). Using (71), (60) and (64) we obtain $M_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_{n_1}$ (n_1 - th column of the $n \times n$ identity matrix)

$$M_{20} = T_{10} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_{n_1-1}, ..., M_{n_1 0} = T_{10}^{n_1-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_1$$

(72) $$M_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_n, M_{02} = T_{01} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_{n-1}, \cdots, M_{0n_2} = T_{01}^{n_2-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_{n_1+1}$$

Note that in this case the matrix

$$[M_{10}, M_{20}, ..., M_{n_1 0}, M_{01}, M_{02}, ..., M_{0n_2}] = [e_{n_1}, e_{n_1-1}, ..., e_1, b_{n_2} e_n, b_{n_2} e_{n-1}, ..., b_{n_2} e_{n_1+1}]$$

is monomial matrix and by the theorem 17 the positive system (57) with (71) and $b_1 = b_2 = \cdots = b_{n_2-1} = 0$, $b_{n_2} \neq 0$ is reachable at the point (n_1, n_2). In the case when $b_k \neq 0$ for $k = 1,...,n_2$ the calculations in the proof are more complicated.

Therefore, the following theorem has been proved.

Theorem 19. Let the internally positive system (57) with (66) be unreachable at the point (n_1, n_2). Then the closed-loop system (68) with (71) is reachable at the point (n_1, n_2) if the state-feedback gain matrix K has the form (70).

Corollary 4. The reachability of internally positive Roesser model (57) with (66) is not invariant under the state-feedback (67).

According the theorem 18 the internally positive system is controllable (to zero) if and only if the matrix A is nilpotent. It is said that the state-feedback (67) violetes the nilpotency of A if and only if the closed-loop matrix (69) is not nilpotent. From theorem 14 the following theorem follows.

Theorem 20. The closed-loop system (68) is uncontrollable at the point (n_1, n_2) if the state-feedback (67) violetes the nilpotency of A.

Corollary 5. The controllability of internally positive Roesser model (57) is not invariant under the state-feedback (69).

5. Concluding remarks.

New notions of the externally and internally positive standard and singular discrete-time and continuous-time linear systems have been introduced. Necessary and sufficient conditions for the external and internal positivity of 1D and 2D linear systems have been established. Sufficient conditions have been also established under which a single-input single-output singular continuous-time system with matrices in canonical forms is internally positive.

It has been shown that:

the reachability and controllability of positive 1D linear systems are not invariant under the state – feedbacks.

for an unreachable (uncontrollable) positive 1D linear system it is possible to choose a suitable state – feedback so that the closed – loop system is reachable (controllable).

the reachability and controllability of positive 2D Roesser model are not invariant under the state – feedbacks.

by suitable choice of the state – feedbacks an unreachable positive 2D Roesser model can be made reachable and a controllable positive 2D Roesser model can be made uncontrollable.

The presented considerations can be easily extended for multi – input 1D and 2D linear systems. It is well – known [25] that the first Fornasini – Marchesini model [9] can be recasted in the 2D Roesser model. Therefore, the considerations can be immediately extended for the positive first Fornasini – Marchesini model. Extensions of the considerations for the positive second Fornasini – Marchesini model [9] and general 2D model [32] are also possible. An open problem is an extension of the considerations for singular 2D linear systems.

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