FACTORING MULTIVARIATE POLYNOMIAL MATRICES

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Abstract

We present an algorithmic proof of Suslin's Stability Theorem, thus providing an algorithm for factoring a multivariate polynomial matrix into a product of elementary matrices. The algorithm involves the use of Gröbner bases, an important tool in computational commutative algebra and computational algebraic geometry. Multivariate polynomial factorization is linked to problems arising in multidimensional digital filter bank design and implementation and in state-space realizations of 2-D systems.

1. Introduction

In 1977, A.A. Suslin proved that $E_m(R)$, the subgroup of the group of invertible matrices which is generated by the elementary matrices, is equal to the special linear group $SL_m(R)$, whenever R is a multivariate polynomial ring over a fixed arbitrary field and m is at least three. (An elementary matrix $e_{ij}(f)$ is a matrix which coincides with the identity except for a single off-diagonal entry f in the ij-position.) One consequence of this result, which is now known as Suslin's Stability Theorem, is that a multivariate polynomial matrix of order at least three with determinant one can be reduced to the identity matrix by means of elementary row and column operations. Another consequence is that such a matrix can be factored into a product of elementary matrices. Over a univariate polynomial ring, such a factorization can be obtained by applying the Euclidean algorithm to any row or column of the matrix. This technique fails, however, if the polynomial ring has at least two variables, since such a polynomial ring is not a Euclidean domain.

The algorithmic version of Suslin's Stability Theorem presented in this paper consists of three key steps:
- a reduction to a special case,
- generation of solutions over finitely many suitable local rings, and
- patching together the local solutions to obtain a global solution.

The algorithm takes as input a matrix A in $SL_m(R)$ and outputs matrices B and C such that $BAC = I_m$, which implies that $A = B^{-1}C^{-1}$. After the reduction step, the algorithm proceeds by induction on the number of variables.

2. Reduction to the Special Case

The first major step is to reduce the input matrix A to a matrix of the form

$$\begin{pmatrix} A' & 0 \\ 0 & I_{m-2} \end{pmatrix},$$

where A' is a 2 by 2 block, and I_{m-2} is the identity of order m - 2. This step is carried out by applying an algorithmic version of the Quillen-Suslin Theorem [3, 4, 5, 9] to one row of the matrix A at a time. The Quillen-Suslin Theorem states that all finitely generated projective modules over a multivariate polynomial ring over a field are free. The algorithmic version used here has as its input a finitely generated module given as the kernel of a unimodular matrix P (in this case a row or column of A) and its output is a matrix U which is a product of elementary matrices such that PU = I; so U gives a free basis for the kernel of P. Thus, for the remainder of the algorithm it suffices to consider square matrices of order three of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with a, b, c, and d multivariate polynomials and $ad - bc = 1$.

3. Generating Local Solutions

The second stage of the algorithm is a local loop which finds factorizations of A over local rings R_{M_i}, with each M_i maximal. The fact that the multivariate polynomial ring R is Noetherian ensures that we pass through this local loop only finitely many times. Working over local rings allows us to carry out some divisions which are not possible in the polynomial ring itself.

It is interesting to note that in practice, it is possible to factor some matrices by working through the local ring algorithm without any divisions, thus obtaining a factorization into polynomial matrices directly. For example, the matrix

$$\begin{pmatrix} 1+\sqrt{2}xy+x^2y^2 & x^4 & 0 \\ y^4 & 1-\sqrt{2}xy+x^2y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

can be directly factored into a product of 109 elementary polynomial matrices by means of the local ring algorithm.

4. Patching Together Local Solutions

Finally in the third stage, the local solutions are lifted to matrices which have polynomial entries and then are patched together in such a way as to obtain a factorization of A over the multivariate polynomial ring itself.

5. Summary

Multivariate polynomial matrices are being viewed as a useful tool in digital signal processing. For example, such matrices are linked to problems arising in the design and implementation of multi-dimensional filter banks. [1, 6] Also, "elementary multivariable polynomial matrices are expected to be an useful tool for obtaining equivalent state-space realizations of possible minimal dimension for a given 2-D linear system." [2] Consequently, algorithms for factoring these matrices are of importance from both a theoretical and a practical viewpoint.

References

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