NORMALIZATION OF TRANSFER MATRIX OF LINEAR SYSTEMS BY FEEDBACKS

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Abstract. The transfer matrix $T(s) = C[I_n s - A]^{-1} B$ of a linear system $\dot{x} = Ax + Bu$, $y = Cx$ can be always written in the standard form $T(s) = \frac{P(s)}{d(s)}$, where $P(s)$ is the polynomial matrix and $d(s)$ is the minimal common denominator. The irreducible transfer matrix is called normal if every nonzero second order minor of $P(s)$ is divisible by $d(s)$. It is shown that for an unnormal transfer matrix of the controllable system there exists a state-feedback gain matrix *K* such that the closed-loop system transfer matrix $T(s) = C[I_n s - (A + BK)]^{-1} B$ is normal. In the case of the output-feedback the closed-loop transfer matrix $T_c(s) = C[I_n s - (A + BFC)]^{-1} B$ can be made normal only if the system is controllable and observable.

Key words: normalization, transfer matrix, feedback, linear system

1. Introduction and preliminaries

Lampe and Rosenwasser in [5,6] have been investigated relationships between the time-domain description and the frequency-domain description. They have shown, for example, that if the normal transfer matrix is written in the standard form $T(s) = \frac{P(s)}{d(s)}$ ($d(s)$ is the minimal common denominator), then every second order nonzero minor of $P(s)$ is divisible by $d(s)$. In [6] it was shown that cyclic matrices are structurally stable i.e. if the matrix $A \in R^{n \times n}$ is cyclic (the minimal polynomial is equal to the characteristic polynomial) and $A_0 \in R^{n \times n}$ is an arbitrary matrix, there exists a positive number ε_0 such that for all $|\varepsilon| < \varepsilon_0$ the matrix $A + \varepsilon A_0$ is cyclic. Some implications of this approach to electrical circuits have been discussed in [2]. The main subject of this paper is to show that for an unnormal transfer matrix of the controllable system there exists a state-feedback gain matrix K such that the closed-loop system transfer matrix $T_c(s) = C[I_n s - (A + BK)]^{-1} B$ is normal. The normalization of transfer matrix by output-feedbacks will be also considered.

Let $R^{m \times n}$ be the set of $m \times n$ real matrices and $R^n := R^{n \times 1}$.

Consider the linear continuous-time system

(1a) $$\dot{x} = Ax + Bu$$

(1b) $$y = Cx$$

where $x = x(t) \in R^n$ is the state vector, $u = u(t) \in R^m$ and $y = y(t) \in R^p$ are the input and output vectors, respectively and $A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}$. It is assumed that $rank B = m$ and $rank C = p$.

The transfer matrix of the system (1) is given by

(2) $$T(s) = C[I_n s - A]^{-1} B$$

which can be written in the standard form

(3) $$T(s) = \frac{P(s)}{d(s)}$$

where $P\in R^{p\times m}[s], R^{p\times m}[s]$ is the set of $p\times m$ polynomial matrices and $d(s)$ is the minimal common denominator of all entries of $T(s)$.

Using elementary row and column operations we may transform any polynomial matrix $P\in R^{p\times m}[s]$ to its Smith canonical form [3,4]

(4) $$P_S(s) = diag\left[i_1(s), i_2(s), ..., i_r(s), 0, ..., 0\right] \in R^{p\times m}[s]$$

where $i_1(s),...,i_r(s)$ are monic invariant polynomials satisfying the divisibility condition $i_{k+1}(s) \mid i_k(s)$, i.e. $i_{k+1}(s)$ is divisible with zero remainder by $i_k(s), k=1,...,r-1$ and $r = rank\, P(s)$.

The invariant polynomials can be determined by the relation

(5) $$i_k(s) = \frac{D_k(s)}{D_{k-1}(s)} \quad (D_0(s)=1),\ k=1,...,r$$

where $D_k(s)$ is the greatest common divisor of all the $k\times k$ minors of $P(s)$.

The characteristic polynomial $\varphi(s) = \det[I_n s - A]$ of the matrix $A\in R^{n\times n}$ and its minimal polynomial $\Psi(s)$ are related by [1]

(6) $$\Psi(s) = \frac{\varphi(s)}{D_{n-1}(s)}$$

From (4)-(6) it follows that $\Psi(s) = \varphi(s)$ if and only if

(7) $$D_1(s) = D_2(s) = \cdots = D_{n-1}(s) = 1$$

A matrix $A\in R^{n\times n}$ satisfying (7) (or equivalently $\Psi(s)=\varphi(s)$) is called cyclic.

2. Normalization of transfer matrix by feedbacks

Consider the system (1) with the state-feedback

(8) $$u = v + Kx,\ \ v\in R^m \text{ and } K\in R^{m\times n}$$

Substitution of (11) into (1a) yields

(9) $$\dot{x} = (A+BK)x + Bv$$

The transfer matrix of the closed-loop system is given by

(10) $$T_c(s) = C[I_n s - (A+BK)]^{-1}B$$

The problem of normalization of transfer matrix by state-feedbacks can be stated as follows. Given the system (1) with A not cyclic and the pair (*A,C*) unobservable. Find a gain matrix K such that the closed-loop transfer matrix (10) is normal.

Theorem 1. Let the matrix A of (1) be not cyclic and the pair (*A,C*) be unobservable. Then there exists a gain matrix K such that the transfer matrix (10) is normal if and only if the pair (*A,B*) is controllable.

Proof. Necessity. It is well-known [3,4] that the pair (*A+BK,B*) is controllable if and only if the pair (*A,B*) is controllable. If the pair (*A,B*) is uncontrollable then the transfer matrix (10) is not normal. Thus if the pair (*A,B*) is uncontrollable then there does not exist K such that the transfer matrix (10) is normal.

Sufficiency. If the pair (*A,B*) is controllable then there exists a non-singular matrix $T\in R^{n\times n}$ such that

(11a) $$\overline{A} = TAT^{-1} = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}, \overline{B} = TB = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix}, A_{ij}\in R^{d_i\times d_j}, B_i\in R^{d_i\times m}$$

where

(11b) $$A_{ij} = \begin{cases} \begin{bmatrix} 0 & I_{d_i-1} \\ \multicolumn{2}{c}{-a_i} \end{bmatrix} & for\quad i=j \\ \begin{bmatrix} 0 \\ -a_{ij} \end{bmatrix} & for\ \ i\neq j \end{cases}, B_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix} \quad \begin{matrix} a_{ij} = [a_0^{ij}\ a_1^{ij} ... a_{d_j-1}^{ij}] \\ b_i = [0\cdots 0\ 1\ b_{i,i+1}\cdots b_{im}] \end{matrix}$$

and $d_1,...,d_m$ are the controllability indexes satisfying $\sum_{i=1}^{m_1} d_i = n$.

Let

(12) $$\hat{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}^{-1} = \begin{bmatrix} 1 & b_{12} & \cdots & b_{1m} \\ 0 & 1 & \cdots & b_{2m} \\ \hdashline 0 & 0 & \cdots & 1 \end{bmatrix}^{-1}$$

and

(13) $$K = \begin{bmatrix} K_1 \\ k \end{bmatrix}, K_1 \in R^{(m-1)\times n}, k = [k_1 \;\; k_2 \cdots k_n] \in R^{1\times n}$$

Using (11a) and (12) it is easy to verify that

(14) $\tilde{B} = \overline{B}\hat{B} = diag[\tilde{b}_1,...,\tilde{b}_m], \tilde{b}_i = [0 \cdots 0 \;\; 1]^T \in R^{d_i}$ (T denotes the transpose)

Define

(15) $$\overline{K} = \hat{B}^{-1}KT^{-1} = \begin{bmatrix} -a_{n_1} + e_{n_1+1} \\ \hdashline -a_{n_{m-1}} + e_{n_{m-1}+1} \\ -a_{n_m} - k \end{bmatrix}$$

where $n_i = \sum_{k=1}^{i} d_k$, a_{n_i} is the n_i-th row of the matrix $\overline{A}, e_i$ is the *i*-th row of I_n.

Using (15), (17) and (18) it is easy to verify that

(16) $$A_c = T(A+BK)T^{-1} = \overline{A} + \overline{B}KT^{-1} = \overline{A} + \tilde{B}\overline{K} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ k_1 & k_2 & k_3 & \cdots & k_n \end{bmatrix}$$

The matrix (16) is cyclic and k will be used to make the pair (A_c, C) observable. It can be shown that if A_c has the Frobenius canonical form (16) then it is always possible to choose its entries $k_1,\cdots,k_n$ so that the pair (A_c, C) is observable. If the matrix A_c is cyclic, the pair (A_c, B) is controllable and the pair (A_c, C) is observable then the transfer matrix (10) is normal.

If the pair (*A*,*B*) is controllable then the gain matrix can be found by the use of the following procedure.

Procedure

Step 1. Compute a non-singular matrix T transforming the pair (*A*,*B*) to the canonical form (11) and $\overline{A}, \overline{B}, \hat{B}, \tilde{B}$.

Step 2. Using (15) compute $\overline{K}$ and

(17) $$K = \hat{B}\overline{K}T$$

with unknown row k.

Step 3. Choose the row vector k so that the pair (A_c, C) is observable

Step 4. To find the desired K substitute k (found in Step 3) into (17).

Now let us consider the system (1) with the output-feedback

(18) $$u = v + Fy, \; F \in R^{m\times p}$$

From (1) and (18) we have

(19) $$\dot{x} = (A + BFC)x + Bv$$

The transfer matrix of the closed-loop system is given by

(20) $$T_c(s) = C[I_n s - (A + BFC)]^{-1} B$$

The problem of normalization of transfer matrix by output-feedbacks can be stated as follows. Given the system (1) with *A* not cyclic, the pair (*A,B*) controllable and the pair (*A,C*) observable. Find a gain matrix *F* such that the closed-loop transfer matrix (20) is normal.

Note that if the pair (*A,C*) is unobservable then the pair (*A+BFC,C*) is also unobservable and the closed-loop transfer matrix (20) is not normal for any gain matrix *F*. Thus, the normalization problem of transfer matrix by output-feedbacks has a solution only if the pair (*A,C*) is observable. If additionally the pair (*A,B*) is controllable the normalization problem is reduced to find a gain matrix such that the closed-loop system matrix $\hat{A}_c = A + BFC$ is cyclic. Let $K = FC$. Then using the approach given in the proof of Theorem 1 we may find *K* given by (17) such that the matrix $\hat{A}_c = A + BK$ is cyclic. By Kronecker – Capelli theorem the equation $K = FC$ has a solution *F* for given *C* and *K* if and only if

(21) $$rank\, C = rank \begin{bmatrix} C \\ K \end{bmatrix}$$

Therefore, the following theorem has been proved.

Theorem 2. Let the pair (*A,B*) be controllable, the pair (*A,C*) be observable and the matrix *A* of (1) be not cyclic. Then there exists a gain matrix *F* such that the transfer matrix (20) is normal if and only if the condition (21) is satisfied.

If (21) holds then applying suitable elementary column operations to *K=FC* we obtain

(22) $$[K_1 \ \ 0] = F[C_1 \ \ 0],\ K_1 \in R^{m\times p}, C_1 \in R^{p\times p}$$

and $\det C_1 \neq 0$ since *C* by assumption has full row rank. From (22) we have

(23) $$F = K_1 C_1^{-1}$$

3. Concluding remarks

It has been shown that for an unnormal transfer matrix of the controllable system (1) there exists the state-feedback (8) such that the closed-loop transfer matrix (10) is normal. In general case the solution to the problem is not unique. A procedure for computation of the state-feedback gain matrix has been given. Necessary and sufficient conditions have been also established for normalization of the transfer matrix of the system (1) by output-feedbacks. With minor modifications the considerations can be also applied to discrete-time linear systems. An extension of these considerations for singular linear systems will be presented in a next paper. An open problem is an extension of there considerations for standard and singular two-dimensional linear systems [3].

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