**SOME RECENT DEVELOPMENTS IN POSITIVE 2D SYSTEMS**

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Necessary and sufficient conditions for the reachability and controllability of positive 2D Roesser model will be established. It will be shown that the reachability and observability of the positive 2D Roesser model will be not invariant under the state-feedbacks. New canonical forms of matrices of singular 2D Roesser model will be introduced. Necessary and sufficient conditions for the existence of a pair of non-singular diagonal matrices transforming the matrices of singular 2D Roesser model to their canonical forms will be established and a procedure for computation of the matrices will be given.

**Main results**

Let $R_+^{n\times m}$ be the set of $n\times m$ real matrices with non-negative entries and $R_+^n := R_+^{n\times 1}$. The set of non-negative integers will be denoted by $Z_+$.

Consider the 2D Roesser model [1]

(1) a) $\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}$ b) $y_{ij} = [C_1 \quad C_2] \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Du_{ij}$, $i,j \in Z_+ := \{0,1,...\}$

where $x_{ij}^h \in R^{n_1}$ and $x_{ij}^v \in R^{n_2}$ are the horizontal and vertical state vectors at the point $(i,j)$, respectively, $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector and $A_{kl} \in R^{n_k \times n_l}, B_k \in R^{n_k \times m}$, $C_k \in R^{p\times n_k}$, $k,l = 1,2$, $D \in R^{p\times m}$.

The Roesser model (1) is called externally positive if for zero boundary conditions $x_{0j}^h = 0, j \in Z_+$, $x_{i0}^v = 0, i \in Z_+$ and all inputs $u_{ij} \in R_+^m$, $i,j \in Z_+$ we have $y_{ij} \in R_+^p$ for $i,j \in Z_+$

**Theorem 1.** The Roesser model (1) is externally positive if and only if its impulse response matrix $g_{ij} \in R_+^{p\times m}$ for $i,j \in Z_+$.

The Roesser model (1) is called internally positive (shortly positive) if for all boundary conditions

(2) $x_{0j}^h \in R_+^{n_1}, j \in Z_+$ and $x_{i0}^v \in R_+^{n_2}, i \in Z_+$

and all $u_{ij} \in R_+^m$, $i,j \in Z_+$ we have $x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \in R_+^n$, $n = n_1 + n_2$ and $y_{ij} \in R_+^p$ for all $i,j \in Z_+$.

**Theorem 2.** The Roesser model (1) is positive if and only if

(3) $$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R_+^{n\times n}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in R_+^{n\times m},\ C = [C_1 \ C_2] \in R_+^{p\times n}, D \in R_+^{p\times m}$$

The transition matrix $T_{ij}$ for (1) is defined as follows [1]

(4) $$T_{ij} = \begin{cases} I_n & for \quad i = j = 0 \\ T_{10}T_{i-1,j} + T_{01}T_{i,j-1} & for \quad i,j \ge 0\,(i+j \ne 0) \\ T_{ij} = 0 & for \quad i < 0 \ \ or/and \ \ j < 0 \end{cases}, \quad T_{10} := \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, T_{01} := \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

From (5) it follows that the transition matrix $T_{ij}$ of the positive model (1) is a positive matrix, $T_{ij} \in R_+^{n\times n}$ for all $i,j \in Z_+$.

The positive Roesser model (1) is called reachable for zero boundary conditions (2) (ZBC) at the point $(h,k),(h,k \in Z_+, h,k > 0)$, if for every $x_f \in R_+^n$ there exists a sequence of inputs $u_{ij} \in R_+^m$ for $(i,j) \in D_{hk}$ such that $x_{hk} = x_f$, where

(5) $$D_{hk} := \begin{Bmatrix} (i,j) \in Z_+ \times Z_+ : 0 \le i \le h, 0 \le j \le k; \\ i + j \ne h + k \end{Bmatrix}$$

The positive Roesser model (1) is called controllable to zero (shortly controllable) at the point $(h,k),(h,k \in Z_+, h,k > 0)$ if for any nonzero boundary conditions

(6) $$x_{0j}^h \in R_+^{n_1},\ 0 \le j \le k \text{ and } x_{i0}^v \in R_+^{n_2},\ 0 \le i \le h$$

there exists a sequence of inputs $u_{ij} \in R_+^m$ for $(i,j) \in D_{hk}$ such that $x_{hk} = 0$.

**Theorem 3.** The positive Roesser model (1) is reachable for ZBC at the point $(h,k)$ if and only if there exists a monomial matrix $R_n$ consisting of *n* linearly independent columns of the reachability matrix

(7) $$R_{hk} := [M_{hk}, M_{h-1,k}, M_{h,k-1}, \ldots, M_{01}, M_{10}],\quad M_{ij} := T_{i-1,j}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{i,j-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

**Theorem 4.** The positive Roesser model (1) is controllable if and only if the matrix *A* is nilpotent matrix, i.e.

(8) $$\det\begin{bmatrix} I_{n_1} z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} z_2 - A_{22} \end{bmatrix} = z_1^{n_1} z_2^{n_2}$$

Consider the Roesser model (1) with the state-feedback

(9) $$u_{ij} = v_{ij} + K\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix},\quad i,j \in Z_+$$

where $K = [K_1, K_2], K_1 \in R^{1\times n_1}, K_2 \in R^{1\times n_2}$ and $v_{ij} \in R^m$ is a new input vector. Substitution of (10) into (1a) yields

(10) $$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = A_c\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Bv_{ij},\quad A_c = A + BK = \begin{bmatrix} A_{11} + B_1K_1, & A_{12} + B_1K_2 \\ A_{21} + B_2K_1, & A_{22} + B_2K_2 \end{bmatrix}$$

The standard closed-loop system (10) is reachable (controllable) if and only if the standard 2D Roesser model (1) is reachable (controllable). It is easy to show that if at least one of $a_l \ne 0, l = 1,\ldots,n_1$ or $b_k \ne 0, k = 1,\ldots,n_2$ then the condition of theorem 3 is not satisfied and the positive model (1) is not reachable at the point $(n_1,n_2)$.

**Theorem 4.** Let the positive system (1) be unreachable at the point $(n_1,n_2)$. Then the closed-loop system (11) is reachable at the point $(n_1,n_2)$ if the state-feedback gain matrix *K* has the form

(11) $$K = [-a_1, -a_2, \ldots, -a_{n_1}, -1, 0, \ldots, 0]$$

The reachability of positive Roesser model (1) is not invariant under the state-feedback (9).

According the theorem 4 the positive system is controllable (to zero) if and only if the matrix *A* is nilpotent.

It is said that the state-feedback (9) violetes the nilpotency of` *A* if and only if the closed-loop $A_c$ is not nilpotent. From theorem 4 the following theorem follows.

**Theorem 5.** The closed-loop system (10) is uncontrollable at the point $(n_1,n_2)$ if the state-feedback (9) violetes the nilpotency of *A*.

The controllability of positive Roesser model (1) is not invariant under the state-feedback (9).

Consider the single–input single–output 2D Roesser model

(12) a) $$E\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = A\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Bu_{ij}$$ , b) $$y_{ij} = C\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}$$ , $i,j \in Z_+$

where $x_{ij}^h \in R^{n_1}$, $x_{ij}^v \in R^{n_2}$, $u_{ij} \in R^m$ and $y_{ij} \in R^m$ are the same as for (1) and

(13) $$E=\begin{bmatrix} E_1 & E_2 \end{bmatrix},\ E_1=\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix},\ E_2=\begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix},\ A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\ C=\begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$$E_{kl}, A_{kl} \in R^{n_k \times n_l},\ B_k \in R^{n_k},\ C_k \in R^{1\times n_k} \text{ for } k,l=1,2$$

It is assumed that $\det E = 0$ and $\det\begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix} \neq 0$ for some $z_1, z_2 \in \mathbb{C}\times\mathbb{C}$

where $\mathbb{C}$ denotes the field of complex numbers.
The transfer matrix of the system (12) is given by

(14) $$T(z_1,z_2)=C\begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix}^{-1} B = \frac{\sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} z_1^{m_1-i} z_2^{m_2-j}}{\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} - a_{ij} z_1^{n_1-i} z_2^{n_2-j}} \quad (m_1 \geq n_1, m_2 \geq n_2)$$

It is said that the matrices (13) have the canonical form if $\overline{E}_{12}=0$, $\overline{E}_{21}=0$,

$$\overline{E}_{11}=\begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{(m_1+1)\times(m_1+1)},\ \overline{E}_{22}=I_{2m_2},\ \overline{A}_{11}=\begin{bmatrix} 0 & I_{m_1} \\ 0 & 0 \end{bmatrix} \in R^{(m_1+1)\times(m_1+1)},$$

$$\overline{A}_{12}=\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in R^{(m_1+1)\times 2m_2},\ \overline{A}_{21}=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \cdots & 0 & 0 \\ a_{n_1 0} & a_{n_1-1,0} & \cdots & a_{00} & 0 \\ a_{n_1 1} & a_{n_1-1,1} & \cdots & a_{01} & 0 \\ \hdashline a_{n_1 n_2} & a_{n_1-1,n_2} & \cdots & a_{0n_2} & 0 \\ b_{m_1 1} & b_{m_1-1,1} & \cdots & b_{11} & b_{01} \\ \hdashline b_{m_1,m_2-1} & b_{m_1-1,m_2-1} & \cdots & b_{1,m_2-1} & b_{0,m_2-1} \\ b_{m_1 m_2} & b_{m_1-1,m_2} & \cdots & b_{1m_2} & b_{0m_2} \end{bmatrix} \in R^{2m_2\times(m_1+1)},$$

(15)

$$\overline{A}_{22}=\left[\begin{array}{cc:cc} 0 & I_{m_2-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & I_{m_2-1} \\ 0 & 0 & 0 & 0 \end{array}\right] \in R^{2m_2\times 2m_2},\ \overline{B}_1=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{m_1+1},\ \overline{B}_2=\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in R^{2m_2},$$

$$\overline{C}_1=\begin{bmatrix} b_{m_1 0} & b_{m_1-1,0} & \cdots & b_{00} \end{bmatrix} \in R^{1\times(m_1+1)},\ \overline{C}_2=\begin{bmatrix} \underbrace{0\cdots 0\ 1}_{m_2+1} & 0\cdots 0 \end{bmatrix} \in R^{1\times 2m_2}.$$

For matrices (13) we shall establish the conditions under which they can be transformed to their canonical forms (15) and we shall find nonsingular matrices

(16) $$P=\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},\ Q=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},\ P_k, Q_k \in R^{n_n \times n_k} \text{ for } k=1,2$$

such that the matrices

(17) $$\overline{E}=\begin{bmatrix} \overline{E}_{11} & 0 \\ 0 & \overline{E}_{22} \end{bmatrix} = P\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}Q = \begin{bmatrix} P_1 E_{11} Q_1 & P_1 E_{12} Q_2 \\ P_2 E_{21} Q_1 & P_2 E_{22} Q_2 \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} = P \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} Q = \begin{bmatrix} P_1 A_{11} Q_1 & P_1 A_{12} Q_2 \\ P_2 A_{21} Q_1 & P_2 A_{22} Q_2 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} = P \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} P_1 B_1 \\ P_2 B_2 \end{bmatrix},$$

$$\overline{C} = [\overline{C}_1 \quad \overline{C}_2] = [C_1 \quad C_2] Q = [C_1 Q_1 \quad C_2 Q_2]$$

have the canonical forms (15).

Necessary conditions and sufficient conditions for the existence of (16) transforming the matrices $E, A, B$ and $C$ to their canonical form (15) and a procedure for computation of the matrices are given in [2].

## References

1. T. Kaczorek, Positive 1D and 2D Systems, Springer-Verlag, London 2002.
2. T. Kaczorek, Canonical forms of singular 1D and 2D linear systems, Intern. J. Applied Math. And Comp. Sci., vol. 13, No 1, 2003 (in press).