A NEW METHOD FOR COMPUTATION OF REALISATIONS IN SINGULAR SYSTEMS

Kaczorek T.

University of Technology, Poland

Abstract. A new method for computation of realizations in singular linear systems is proposed. The proposed method is based on a transformation of improper transfer matrices to equivalent proper transfer matrices. A procedure is derived for computation of realizations of a given improper transfer matrix. The procedure is illustrated by a numerical example.

1. Introduction.

The realization problem for standard and singular 1D and 2D linear systems has been considered in many papers and books [1,2,4-7]. The realization problem has been also investigated for positive 1D and 2D systems [1,3].

In this paper a new method for computation of realizations in singular linear systems will be proposed. The method is based on a transformation of improper transfer matrices to equivalent proper transfer matrices for which realizations can be found by the use of the well-known methods. A procedure for computation of realizations of a given improper transfer matrix will be derived and illustrated by a numerical example.

2. Problem formulation.

Let $R^{m\times n}$ be the set of $m\times n$ real matrices and $R^n := R^{n\times 1}$. Consider the singular linear time-continuous system

$$E\dot{x} = Ax + B_0 u + B_1 \dot{u} \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where $x\in R^n$, $u\in R^m$ and $y\in R^p$ are the state, input and output vectors, respectively and

$E, A\in R^{n\times n}, B_0, B_1\in R^{n\times m}, C\in R^{p\times n}, D\in R^{p\times m}$.

It is assumed that $\det E = 0$ but the pair (*E,A*) is regular, i.e.

$\det[Es - A] \neq 0$ for some $s\in \mathbf{C}$

where **C** is the field of comlex numbers.

The transfer matrix of (1) is given by

$$T(s) = C[Es - A]^{-1}(B_0 + sB_1) + D \tag{3}$$

The transfer matrix (3) is called proper (strictly proper) if and only if

$\lim_{s\to\infty} T(s) = K \in R^{p\times m}$ and $K \neq 0 \quad (K = 0)$ (4)

otherwise it is called improper.

The equations (1) can be written as

$$\overline{E}\dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u \tag{5a}$$

$$y = \overline{C}x + \overline{D}u \tag{5b}$$

where

$$\overline{E} = \begin{bmatrix} E, & -B_1 \\ 0, & 0 \end{bmatrix} \in R^{\overline{n}\times\overline{n}}, \overline{x} = \begin{bmatrix} x \\ u \end{bmatrix} \in R^{\overline{n}},$$

$$\overline{A} = \begin{bmatrix} A & 0 \\ 0 & I_m \end{bmatrix} \in R^{\overline{n}\times\overline{n}}, \overline{B} = \begin{bmatrix} B_0 \\ -I_m \end{bmatrix} \in R^{\overline{n}\times m}, \overline{n} = n + m \tag{5c}$$

$$\overline{C} = [C \quad 0] \in R^{p\times\overline{n}}, \overline{D} = D$$

The equations (1) can be also written in the form

$$\overline{E}\dot{\overline{x}} = \widetilde{A}\overline{x} + \widetilde{B}u \tag{6a}$$

$$y = \widetilde{C}\overline{x} \tag{6b}$$

where

$$\widetilde{A} = \begin{bmatrix} A & B_0 \\ 0 & I_m \end{bmatrix} \in R^{\bar{n}\times\bar{n}}, \widetilde{B} = \begin{bmatrix} 0 \\ -I_m \end{bmatrix} \in R^{\bar{n}\times m}, \quad (6c)$$
$$\widetilde{C} = [C \quad D] \in R^{p\times\bar{n}}$$

The matrices E, A, B_0, B_1, C and D are called a realization of a given rational matrix $T(s) \in R^{p\times m}(s)$ (the set of $p\times m$ rational matrices in s) if they satisfy the equality (3). The realization (E, A, B_0, B_1, C, D) is called minimal if and only if the matrices E and A have minimal dimensions amongs all realizations of $T(s)$.

The realization problem under the considerations can be stated as follows. Given an improper rational matrix $T(s) \in R^{p\times m}(s)$, find its realization (E, A, B_0, B_1, C, D).

3. Solution of the problem.

Any rational matrix $T(s) \in R^{p\times m}(s)$ can be written in the form

$$T(s) = \frac{P(s)}{d(s)} \quad (7)$$

where $P(s)$ is the $p\times m$ polynomial matrix and

$$d(s) = d_q s^q + d_{q-1}s^{q-1} + \cdots + d_1 s + d_0 \quad (8)$$

is the least common denominator.

Let $N = \deg P(s)$ be the degree of the polynomial matrix $P(s)$ and $N > q$.

The proposed method is based on the following theorem.

Theorem 1. Let $s = \omega^{-1} + \lambda$ and $d(\lambda) \neq 0$ for the improper matrix (7), $N > q$. Then the rational matrix in ω

$$\overline{T}(\omega) = T(s)_{|s=\varpi^{-1}+\lambda} = \frac{\overline{P}(\omega)}{\overline{d}(\omega)} \quad (9)$$

is proper i.e. $\deg \overline{d}(\omega) = N \geq \deg \overline{P}(\omega)$.

Proof. Substitution of $s = \omega^{-1} + \lambda$ into $T(s)$ yields the improper matrix in ω^{-1}

$$T(\omega^{-1} + \lambda) = \frac{P(\omega^{-1} + \lambda)}{d(\omega^{-1} + \lambda)} \quad (10)$$

since the degree of $P(\omega^{-1} + \lambda)$ and $d(\omega^{-1} + \lambda)$ with respect to ω^{-1} is equal to N and q, respectively. Multiplying the numerator and denominator of (10) by ω^N we obtain (9) with $\deg \overline{d}(\omega) = N \geq \deg \overline{P}(\omega)$ since by assumption $d(\lambda) \neq 0$.

Note that the theorem allow us to reduce the problem of computation of a realization (E, A, B_0, B_1, C, D) of the improper matrix $T(s)$ to the problem of computation of a realization $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ of the proper matrix $\overline{T}(\omega)$.

Using one of the well-known methods [4,5] we can find the realization $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ of the given proper matrix $\overline{T}(\omega)$.

Let us define

$$\begin{aligned} &E = A_\varpi, A = I_n + \lambda A_\varpi, B_0 = \lambda B_\varpi, \\ &B_1 = -B_\varpi, C = C_\varpi, D = D_\varpi \end{aligned} \quad (11)$$

Substituting (11) and $s = \Omega^{-1} + \lambda$ into (3) we obtain

$$\begin{aligned} &T(s) = C[Es - A]^{-1}(B_0 - sB_1) + D = \\ &= C_\varpi[A_\varpi(\omega^{-1} + \lambda) - (I_n + \lambda A_\varpi)]^{-1}(\lambda B_\varpi - B_\varpi s) + D_\varpi = \\ &= C_\varpi[A_\varpi \omega^{-1} - I_n]^{-1}(\lambda - s)B_\varpi + D_\varpi = \\ &= C_\varpi[I_n \omega - A_\omega]^{-1} B_\varpi + D_\varpi \end{aligned} \quad (12)$$

since $\omega^{-1} = s - \lambda$.

Therefore, the following theorem has been proved.

Theorem 2. If $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a realization of $\overline{T}(\omega)$ given by (9) then (E, A, B_0, B_1, C, D) defined by (11) is a realization of $T(s)$.

From the above considerations the following procedure for computation of the realization (E, A, B_0, B_1, C, D) of $T(s)$ follows.

Procedure

Step 1. Write the given transfer matrix $T(s)$ in the form (7) and choose a scalar λ so that $d(\lambda) \neq 0$.

Step 2. Substituting $s = \omega^{-1} + \lambda$ into $T(s)$ and multiplying the numerator and the denumerator of (10) by ω^N find the matrix $\overline{T}(\omega)$.

Step 3. Using one of the well-known methods [1,5] find a realization $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ of the matrix $\overline{T}(\omega)$.

Step 4. Using (11) find the desired realization (E, A, B_0, B_1, C, D) of the matrix $T(s)$.

Remark 1. For two different values of λ we obtain in general case different realizations $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ and (E, A, B_0, B_1, C, D), respectively.

Remark 2. If $d(0) \neq 0$ the choice $\lambda = 0$ is recommended. In this case from (11) we have

$$E = A_\varpi, A = I_n, B_0 = 0, B_1 = -B_\varpi, C = C_\varpi, D = D_\varpi \quad (11')$$

Using the procedures we shall find a realization (E, A, B_0, B_1, C, D) of the transfer function

$$T(s) = \frac{a_N s^N + \cdots + a_1 s + a_0}{s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0} \text{ with } N > q \quad (13)$$

Step 1. In this case we have

$$P(s) = a_N s^N + \cdots + a_1 s + a_0$$
$$(14)$$
$$d(s) = s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0$$

and we choose λ so that $d(\lambda) \neq 0$.

Step 2. Substitution of $s = \omega^{-1} + \lambda$ into (13) yields

$$T(\omega^{-1} + \lambda) = \frac{a_N (\omega^{-1} + \lambda)^N + \cdots + a_1 (\omega^{-1} + \lambda) + a_0}{(\omega^{-1} + \lambda)^q + b_{q-1} (\omega^{-1} + \lambda)^{q-1} + \cdots + b_1 (\omega^{-1} + \lambda) + b_0} \quad (15)$$

and after multiplication of the numerator and the denominator of (15) by ω^N we obtain

$$\overline{T}(\omega) = \frac{a_0 \omega^N + \cdots + a_N}{b_0 \omega^N + \cdots + \omega^{N-q}} =$$
$$(16)$$
$$= \frac{a_0}{b_0} + \frac{\overline{a}_{N-1} \omega^{N-1} + \cdots + \overline{a}_1 \omega + \overline{a}_0}{\omega^N + b_0 \omega^{N-1} + \cdots + \omega^{N-q}}$$

Step 3. The well-known [4,5] realization of (16) has the form

$$A_\varpi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 \\ 0 & \cdots & 0-1 & \cdots & -b_1, -b_0 \end{bmatrix} \in R^{N \times N},$$

$$B_\varpi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^N, C_\varpi = [\overline{a}_0 \ \overline{a}_1 \ldots \overline{a}_{N-1}], D_\varpi = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \quad (17)$$

Note that if $N > q$ then $\det A_\varpi = 0$ and the matrix *E* is singular.

Step 4. Using (11) and (17) we obtain the desired realization (E, A, B_0, B_1, C, D) of the transfer function (13).

Example. Find two realizations (E, A, B_0, B_1, C, D) of the transfer function

$$T(s) = \frac{s^2+2s+3}{s+1} \tag{18}$$

Using the procedure for two different choice of the scalar λ we obtain

Step 1. In this case

$$P(s) = s^2+2s+3 \text{ and } d(s) = s+1$$

We choose $\lambda = 0$ and $\lambda = 1$ since $d(0) = 1$ and $d(1) = 2$.

Step 2. Substitution of $s = \omega^{-1}$ and $s = \omega^{-1}+1$ into (18) yields

$$T(\omega^{-1}) = \frac{\omega^{-2}+2\omega^{-1}+3}{\omega^{-1}+1} \tag{19a}$$

and

$$T(\omega^{-1}+1) = \frac{\omega^{-2}+4\omega^{-1}+6}{\omega^{-1}+2} \tag{19b}$$

respectively.

Multiplying the numerator and the denominator of (19) by ϖ^2 we obtain

$$\overline{T}_1(\omega) = \frac{3\omega^2+2\omega+1}{\omega^2+\omega} = 3 + \frac{-\omega+1}{\omega^2+\omega}$$

and (20)

$$\overline{T}_2(\omega) = \frac{6\omega^2+4\omega+1}{2\omega^2+\omega} = 3 + \frac{\frac{1}{2}\omega+\frac{1}{2}}{\omega^2+\frac{1}{2}\omega}$$

Step 3. The realizations of $\overline{T}_1(\omega)$ and $\overline{T}_2(\omega)$ are given respectively by

$$A_\varpi^1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, B_\varpi^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_\varpi^1 = [1,-1], D_\varpi^1 = [3] \tag{21}$$

and

$$A_\varpi^2 = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}, B_\varpi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_\varpi^2 = \left[\frac{1}{2}, \frac{1}{2}\right], D_\varpi^2 = [3] \tag{22}$$

Step 4. Using (11) and (21), (22) we obtain the desired realizations of (18) in the forms

$$\begin{gathered} E_1 = A_\varpi^1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ B_0^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_1^1 = -B_\varpi^1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ C_1 = C_\varpi^1 = [1,-1], D_1 = D_\varpi^1 = [3] \end{gathered} \tag{23}$$

and

$$\begin{gathered} E_2 = A_\varpi^2 = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}, A_2 = I_n + \lambda A_\varpi^2 = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \\ B_0^2 = \lambda B_\varpi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_1^2 = -B_\varpi^2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ C_2 = C_\varpi^2 = \left[\frac{1}{2}, \frac{1}{2}\right], D_2 = D_\varpi^2 = [3] \end{gathered} \tag{24}$$

It is easy to check that (23) and (24) are realizations of (18).

Theorem 3. The singular system (5) is completely controllable and completely observable if the pair (A_ϖ, B_ϖ) is controllable and the pair (A_ϖ, C_ϖ) is observable.

Proof. To prove the complete controllability of the system (5) we have to show that

$$rank\left[\overline{E}, \overline{B}\right] = \overline{n} \tag{25a}$$

and

$$rank\left[\overline{E}s - \overline{A}, \overline{B}\right] = \overline{n} \text{ for all finite } s \in \mathbf{C} \tag{25b}$$

The details of the proof will be given for single-input (m=1) and single-output (p=1) systems. Without loss of generality it can be assumed that the matrices A_ϖ, B_ϖ i C_ϖ have the forms (17).

Using (5c) and (11) it is easy to show that the condition (25a) is satisfied since

$$rank[\overline{E},\overline{B}] = rank\begin{bmatrix} E & -B_1 & \vdots & B_0 \\ 0 & 0 & \vdots & -I_m \end{bmatrix} = \qquad (26a)$$

$$= rank\begin{bmatrix} A_\sigma & -B_\varpi & \vdots & \lambda B_\varpi \\ 0 & 0 & \vdots & -I_m \end{bmatrix} =$$

$$rank\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0-1 & \cdots & -b_0 & 1 & \lambda \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix} = N = \overline{n}$$

The condition (25b) is also satisfied since

$$rank\left[\overline{E}s - \overline{A}, \overline{B}\right] = rank\begin{bmatrix} Es - A & -B_1 s & \vdots & B_0 \\ 0 & -I_m & \vdots & -I_m \end{bmatrix} =$$

$$= rank\begin{bmatrix} A_\omega s - (I_n + \lambda A_\varpi) & B_\varpi s & \vdots & \lambda B_\varpi \\ 0 & -I_m & \vdots & -I_m \end{bmatrix} =$$

$$rank\begin{bmatrix} -1 & s-\lambda & 0 & \cdots & 0 & 0 & \vdots & 0 \\ 0 & -1 & s-\lambda & \cdots & 0 & 0 & \vdots & 0 \\ \hdashline 0 & 0 & 0 & \cdots & -1, s-\lambda & \vdots & \vdots & \vdots \\ 0 & \cdots & s-\lambda & \cdots & b_0(s-\lambda)-1 & -s & \vdots & \lambda \\ 0 & 0 & \cdots & \cdots & 0 & -1 & \vdots & -1 \end{bmatrix} = N = \overline{n}$$

for all finite $s \in \mathbf{C}$ (26b)

Likewise, to prove the complete observability of the system (5) we have to show that

$$rank\begin{bmatrix} \overline{E} \\ \overline{C} \end{bmatrix} = \overline{n} \qquad (27a)$$

and

$$rank\begin{bmatrix} \overline{E}s - \overline{A} \\ \overline{C} \end{bmatrix} = \overline{n} \text{ for all finite } s \in \mathbf{C} \qquad (27b)$$

Using (5c) and (11) it is easy to show that the condition (27a) is satisfied since

$$rank\begin{bmatrix} \overline{E} \\ \overline{C} \end{bmatrix} = rank\begin{bmatrix} E & -B_1 \\ 0 & 0 \\ C & 0 \end{bmatrix} = rank\begin{bmatrix} A_\varpi & B_\varpi \\ 0 & 0 \\ C_\varpi & 0 \end{bmatrix} =$$

$$rank\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & \vdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \vdots & 0 \\ & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \vdots & 0 \\ 0 & 0 & -1 & \cdots & -b_0 & \vdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & \vdots & 0 \\ \overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \cdots & \overline{a}_{N-1} & \vdots & 0 \end{bmatrix} = N = \overline{n}$$

The condition (27b) is also satisfied since

$$rank\begin{bmatrix} \overline{E}s-\overline{A} \\ \overline{C} \end{bmatrix} = rank\begin{bmatrix} Es-A & -B_1 s \\ 0 & -I_m \\ C & 0 \end{bmatrix} =$$

$$= rank\begin{bmatrix} A_\varpi s-(I_n+\lambda A_\varpi) & -B_\varpi s \\ 0 & -I_m \\ C_\varpi & 0 \end{bmatrix} =$$

$$rank\left[\begin{array}{ccccc:c} -1 & s-\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & s-\lambda & \cdots & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & -1, s-\lambda & 0 \\ 0 & \cdots & s-\lambda & \cdots & b_0(s-\lambda)-1 & s \\ 0 & 0 & \cdots & \cdots & 0,0,0 & -1 \\ \overline{a}_0 & \overline{a}_1 & \cdots & \cdots & \overline{a}_{N-2,}\overline{a}_{N-1} & 0 \end{array}\right] = N = \overline{n}$$

Remark 3. In a similar way it can be proved that the system (6) is completely controllable and completely observable if the pair (A_ϖ, B_ϖ) is controllable and the pair (A_ϖ, C_ϖ) is observable.

Remarks 4. The realization (E, A, B_0, B_1, C, D) with (11) is minimal since it is completely controllable and completely observable.

4. Concluding remarks

A new method for computation of realizations in singular continuous-time linear systems has been proposed. The method is based on the transformation of improper transfer matrices to equivalent proper transfer matrices for which realizations can be computed by the use of the well-known methods. A procedure for computation of realizations of a given improper transfer matrix has been derived and illustrated by a numerical example.

Knowing the realization (E, A, B_0, B_1, C, D) and using (5c) we may find the realization $(\overline{E}, \overline{A}, \overline{B}, \overline{C}, \overline{D})$ of a given improper transfer matrix $T(s)$. It has been shown that the singular system (5) is completely controllable and completely observable if the pair (A_ϖ, B_ϖ) is controllable and the pair (A_ϖ, C_ϖ) is observable. With minor modifications (the variable s should be replaced by the variable *z*) the method can be also applied for computation of realization in singular discrete-time linear systems.The considerations can be also extended for singular 2D linear systems [7,2,4].

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