PHASE SHIFTING IN THE QUINCUNX FILTER BANK

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Abstract

In this paper, a recursive method for recovering the undecimated wavelet coefficients from the decimated wavelet coefficients resulting from a nonseparable perfect reconstruction filter bank is given. The method is applied to the quincunx filter bank based on the lifting scheme.

1. Introduction

Multirate filtering is a powerful tool for obtaining more compact signal representations. In particular, efficient wavelet filters based on separable sampling grids have been widely adopted in image compression tasks. Additional improvement in terms of energy compaction could be obtained by non-separable, e.g. quincunx sampling grids. However, both separable and non-separable schemes utilize critical sampling to achieve non-redundant image representation, thus introducing aliasing and sacrificing the phase information. In many applications, such as feature extraction and denoising, the phase information is vitally important. It can be retained by an undecimated filter bank, leading to an overcomplete expansion, which is, however, redundant and less efficient for compression. Having these restrictions in mind, an attractive alternative is to design a tool aimed at recovering all missing phases (coefficients) out of the critically sampled coefficients.

In this paper a method for recovering the missing phases on the quincunx sampling grid is proposed as a generalization of the 1D method given in 1. Such a method is computationally more efficient than the straightforward low-band-shift method 2. The proposed method is applied to a quincunx filter bank based on the lifting scheme.

2. Perfect Reconstruction Filter Bank Based On Quincunx Sampling

Two-dimensional downsampling on a lattice defined by the subsampling matrix $\mathbf{D}$ can be expressed in the z -domain as

$$Y_D(\mathbf{z}) = (\downarrow \mathbf{D})X(\mathbf{z}) = \frac{1}{|\det \mathbf{D}|} \sum_{k \in N(\mathbf{D}^T)} X\left(\mathbf{z}^{\mathbf{D}^{-1}} \circ e^{-j2\pi \mathbf{k}^T \mathbf{D}^{-1}}\right), \tag{1}$$

where $\mathbf{z} = [z_1\ z_2]^T$, and $N(\mathbf{D}^T)$ is a set of integer vectors in the parallelepiped defined by the columns of the transpose of the subsampling matrix $\mathbf{D}$ 3. For the quincunx case,

$$Y_D(\mathbf{z}) = \frac{1}{2}\left[X\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) + X\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right)\right] = \frac{1}{2}\left[X\left(z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}, z_1^{-\frac{1}{2}} z_2^{\frac{1}{2}}\right) + X\left(-z_1^{\frac{1}{2}} z_2^{\frac{1}{2}}, -z_1^{-\frac{1}{2}} z_2^{\frac{1}{2}}\right)\right], \text{ where } \mathbf{D} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{2}$$

Since the determinant of the quincunx subsampling matrix equals 2, the perfect reconstruction filter bank requires two analysis filters only: the low-pass channel filter H_0 and the high-pass channel filter H_1. The undecimated low-pass filter output, $A(\mathbf{z}) = H_0(\mathbf{z})X(\mathbf{z})$ consist of two phases:

$$A_0(\mathbf{z}) = (\downarrow \mathbf{D})H_0(\mathbf{z})X(\mathbf{z}), \qquad A_1(\mathbf{z}) = (\downarrow \mathbf{D})\mathbf{z}^{-\mathbf{t}} H_0(\mathbf{z})X(\mathbf{z}), \tag{3}$$

where $\mathbf{t} = [1\ 0]^T$, and $\mathbf{z}^{-\mathbf{t}} = \mathbf{z}_1^{-1}$. By using the relation (2):

$$\begin{aligned} A_0(\mathbf{z}) &= \tfrac{1}{2}\left[H_0\left(\mathbf{z}^{\mathbf{D}^{-1}}\right)X\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) + H_0\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right)X\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right)\right] \\ A_1(\mathbf{z}) &= \tfrac{1}{2}\mathbf{z}^{-\mathbf{u}}\left[H_0\left(\mathbf{z}^{\mathbf{D}^{-1}}\right)X\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) - H_0\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right)X\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right)\right], \text{ where } \mathbf{z}^{-\mathbf{u}} = \left(\mathbf{z}^{\mathbf{D}^{-1}}\right)^{-\mathbf{t}} = z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \end{aligned} \tag{4}$$

Similar equations can also be written for the high-pass channel. In the matrix form, both the low-pass and high-pass equations can be written as

$$\begin{bmatrix} A_0(\mathbf{z}) \\ D_0(\mathbf{z}) \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} H_0\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) & H_0\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) \\ H_1\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) & H_1\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) \end{bmatrix}}_{\mathbf{H}_{m0}(\mathbf{z})} \begin{bmatrix} X\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) \\ X\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) \end{bmatrix},$$

$$\begin{bmatrix} A_1(\mathbf{z}) \\ D_1(\mathbf{z}) \end{bmatrix} = \frac{1}{2} \mathbf{z}^{-\mathbf{u}} \underbrace{\begin{bmatrix} H_0\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) & -H_0\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) \\ H_1\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) & -H_1\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) \end{bmatrix}}_{\mathbf{H}_{m1}(\mathbf{z})} \begin{bmatrix} X\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) \\ X\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) \end{bmatrix}. \tag{5}$$

3. Recovering the Missing Phase

Relationships between the two phases can be expressed as

$$\begin{bmatrix} A_1(\mathbf{z}) \\ D_1(\mathbf{z}) \end{bmatrix} = \mathbf{z}^{-\mathbf{u}} \underbrace{\mathbf{H}_{m1}(\mathbf{z})\mathbf{H}_{m0}^{-1}(\mathbf{z})}_{\mathbf{T}_1(\mathbf{z})} \begin{bmatrix} A_0(\mathbf{z}) \\ D_0(\mathbf{z}) \end{bmatrix}, \qquad \begin{bmatrix} A_0(\mathbf{z}) \\ D_0(\mathbf{z}) \end{bmatrix} = \mathbf{z}^{-\mathbf{u}} \underbrace{\mathbf{H}_{m0}(\mathbf{z})\mathbf{H}_{m1}^{-1}(\mathbf{z})}_{\mathbf{T}_2(\mathbf{z})} \begin{bmatrix} A_1(\mathbf{z}) \\ D_1(\mathbf{z}) \end{bmatrix}. \tag{6}$$

It can be seen that the average or detail image from the odd phase can be obtained based on the average and detail images from the even phase and vice versa. The phase shifting is performed using the phase shifting matrices $\mathbf{T}_1(\mathbf{z})$ and $\mathbf{T}_2(\mathbf{z})$, being products of the matrices $\mathbf{H}_{m0}$, $\mathbf{H}_{m1}$, and their inverses, as shown in Eq. (6). It is easy to prove that $\mathbf{T}_1(\mathbf{z}) = \mathbf{T}_2(\mathbf{z}) = \mathbf{T}(\mathbf{z})$. Since the filter bank satisfies the perfect reconstruction condition, $det(\mathbf{H}_{m0}(\mathbf{z})) = |\det(\mathbf{D})| = 2$, so we obtain

$$\mathbf{T} = \frac{1}{2}\begin{bmatrix} T_{11}(\mathbf{z}) & T_{12}(\mathbf{z}) \\ T_{21}(\mathbf{z}) & T_{22}(\mathbf{z}) \end{bmatrix}, \text{ where} \begin{cases} T_{11}(\mathbf{z}) = H_0\left(\mathbf{z}^{\mathbf{D}^{-1}}\right)H_1\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) + H_0\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right)H_1\left(\mathbf{z}^{\mathbf{D}^{-1}}\right), \\ T_{12}(\mathbf{z}) = -2H_0\left(\mathbf{z}^{\mathbf{D}^{-1}}\right)H_0\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right), \\ T_{21}(\mathbf{z}) = 2H_1\left(\mathbf{z}^{\mathbf{D}^{-1}}\right)H_1\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right), \\ T_{22}(\mathbf{z}) = -H_0\left(\mathbf{z}^{\mathbf{D}^{-1}}\right)H_1\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right) - H_0\left(-\mathbf{z}^{\mathbf{D}^{-1}}\right)H_1\left(\mathbf{z}^{\mathbf{D}^{-1}}\right) = -T_{11}(\mathbf{z}). \end{cases} \tag{7}$$

4. Results for the Lifting Scheme

The so obtained results are applied to the quincunx filter bank based on the lifting scheme 4, shown in Fig. 1. An example of quincunx predict and update filters is given in 5. The corresponding low-pass and high-pass analysis filters are obtained as:

$$\begin{aligned} H_0(\mathbf{z}) &= 1 - P(\mathbf{z}^{\mathbf{D}})U(\mathbf{z}^{\mathbf{D}}) + \mathbf{z}^{-\mathbf{t}}U(\mathbf{z}^{\mathbf{D}}), \\ H_1(\mathbf{z}) &= -P(\mathbf{z}^{\mathbf{D}}) + \mathbf{z}^{-\mathbf{t}}. \end{aligned} \tag{8}$$

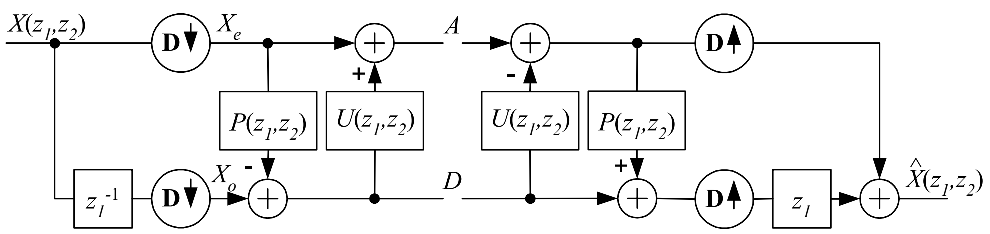

Figure 1. Quincunx filter bank based on the lifting scheme.

Now, the elements of the phase shifting matrix can be expressed as

$$\begin{aligned} T_{11}(\mathbf{z}) &= 2\left[P(\mathbf{z})\left(P(\mathbf{z})U(\mathbf{z}) - 1\right) - z_1^{-1}z_2^{-1}U(\mathbf{z})\right], \\ T_{12}(\mathbf{z}) &= -2\left[\left(P(\mathbf{z})U(\mathbf{z}) - 1\right)^2 - z_1^{-1}z_2^{-1}U^2(\mathbf{z})\right], \\ T_{21}(\mathbf{z}) &= 2\left(P^2(\mathbf{z}) - z_1^{-1}z_2^{-1}\right). \end{aligned} \tag{9}$$

5. Recovering All Phases in the *N*-th Level

For an N-level wavelet decomposition, recursive calculation can be used to obtain all the 2^N phases for a given decomposition level. In the critically sampled filter bank, only the zero-th phases of wavelet coefficients are known for each decomposition level. To recover the remaining 2^N-1 phases in the N-th decomposition level, we start by calculating the first phase coefficients from the zero-th phase coefficients by using the relation (6).

After that, we find the next two phases (phase 2 and 3) by using the relation (6) for the level N-1. In the following equations, subscripts of A and D denote decomposition level, while their superscripts denote phase number.

$$\begin{aligned} D_N^2(\mathbf{z}^{\mathbf{D}}) + \mathbf{z}^{\mathbf{t}} D_N^3(\mathbf{z}^{\mathbf{D}}) &= H_1(\mathbf{z}) A_{N-1}^1(\mathbf{z}) = H_1(\mathbf{z})\mathbf{z}^{-\mathbf{u}}\left[T_{11}(\mathbf{z})A_{N-1}^0(\mathbf{z}) + T_{12}(\mathbf{z})D_{N-1}^0(\mathbf{z})\right] \\ &= \mathbf{z}^{-\mathbf{u}}\left[T_{11}(\mathbf{z})\left[D_N^0(\mathbf{z}^{\mathbf{D}}) + \mathbf{z}^{\mathbf{t}} D_N^1(\mathbf{z}^{\mathbf{D}})\right] + H_1(\mathbf{z})T_{12}(\mathbf{z})D_{N-1}^0(\mathbf{z})\right], \end{aligned} \tag{10}$$

where $\mathbf{u} = \mathbf{D}^{-1}\mathbf{t} = [0.5\ 0.5]^T$. Similar equation can be derived for the average coefficients.

The subsequent step is to recover the next four phases (phase 4, 5, 6 and 7) of wavelet coefficients of the N-th decomposition level by using the relation (6) for the level N-2. Of course, if necessary, each phase can be separately extracted by using decimation preceded with the appropriate shift.

To generalize, phases 2^k to $2^{k+1}-1$ in the N-th decomposition level, can be recursively recovered based on previous phases 0 to 2^k-1 by using the relation (6) for the level $N-k$.

References

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