# STRUCTURE DECOMPOSITION AND COMPUTATION OF MINIMAL REALIZATION OF NORMAL TRANSFER MATRIX OF POSITIVE SYSTEMS

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**Abstract:** The notions of the normal matrix and of the structure decomposition of normal matrices are extended for linear standard positive systems. It is shown that there exists the structure decomposition of a transfer matrix if and only if the transfer matrix is normal. A procedure for computation of the structure decomposition is derived. The structure decomposition is applied for establishing the existence conditions and for the computation of minimal positive realizations of transfer matrices. A procedure for computation of minimal positive realizations with the matrix $\mathbf{A}_J$ in the Jordan form is presented and illustrated by a numerical example.

## 1. Introduction

Lampe and Rosenwasser in [7, 8] have introduced the notions of the normal matrix and the structure decomposition ($S-$Darstellung) of normal matrices. They have shown that if the normal transfer matrix is written in the standard form $\mathbf{T}=\frac{\mathbf{N}}{d}$ ($d$ is the minimal common denominator) then every second order nonzero minor of the polynomial matrix $\mathbf{N}$ is divisible (with zero remainder) by the polynomial $d$. They have also shown that there exists a structure decomposition of transfer matrices if and only if the matrices are normal. The influence of the state-feedback in cyclity of linear systems and the normalization of linear systems by state-feedbacks have been considered in [3, 4]. Some implications of the notion of the normal matrix on electrical circuit have been discussed in [6].

The realization problem for positive linear systems has been considered in many papers and books [1, 2, 5].

In this paper the notions of the normal matrix and of the structure decomposition of normal matrices will be extended for standard positive systems. The structure decomposition will be applied for establishing the existence conditions and for the computation of minimal positive realizations with the matrix $\mathbf{A}_J$ in the Jordan form. To the best author knowledge the above problems for linear positive systems have been not considered yet.

## 2. Preliminaries

Let $R^{m\times n}$ be the set of $m\times n$ real matrices and $R^m = R^{m\times 1}$. Consider the discrete-time linear system

(1a) $$x_{i+1} = \mathbf{A}x_i + \mathbf{B}u_i$$

(1b) $$y_i = \mathbf{C}x_i + \mathbf{D}u_i$$

$i\in Z_+ = \{0,1,\dots\}$

where $x_i\in R^n$, $u_i\in R^m$ and $y_i\in R^p$ are the state, input and output vectors, respectively and $\mathbf{A}\in R^{n\times n}$, $\mathbf{B}\in R^{n\times m}$, $\mathbf{C}\in R^{p\times n}$, $\mathbf{D}\in R^{p\times m}$.

Let $R_+^{m\times n}$ be the set of $m\times n$ real matrices with nonnegative entries and $R_+^n = R_+^{n\times 1}$. The system (1) is called (internally) positive if and only if for every $x_0\in R_+^n$ and any input sequence $u_i\in R_+^m$, $i\in Z_+$, we have $x_i\in R_+^n$ and $y_i\in R_+^p$ for all $i\in Z_+$.

**Theorem 1**. [2, 5]. *The system (1) is positive if and only if*

(2) $$\mathbf{A}\in R_+^{n\times n},\ \mathbf{B}\in R_+^{n\times m},\ \mathbf{C}\in R_+^{p\times n},\ \mathbf{D}\in R_+^{p\times m}$$

The transfer matrix of (1) is given by

(3) $$\mathbf{T}(z)=\mathbf{C}\left[\mathbf{I}_n z-\mathbf{A}\right]^{-1}\mathbf{B}+\mathbf{D}\in R^{p\times m}(s)$$ (the set of $p\times m[0,1]$ rational matrix)

which can be written in the form

(4) $$\mathbf{T}(z)=\frac{\mathbf{N}(z)}{d(z)}$$

where $\mathbf{N}(z)\in R_+^{p\times m}[z]$ (the set of $p\times m$ polynomial matrices) and $d(z)$ is the minimal common denominator.

The matrix $\mathbf{T}(z)$ is irreducible if and only if for any zero $z_i$ of $d(z)$ ($d(z_i)=0$) we have $\mathbf{N}(z_i)\neq 0$.

The matrix (4) has the standard form if and only if is irreducible and the polynomial $d(z)$ is a monic polynomial ,i.e. the coefficient at the highest degree of $z$ is equal to $1$.

**Definition 1**. *The standard matrix (4) with* $\min\left(p,m\right)\geq 2$ *is called normal if and only if every nonzero second order minor of the polynomial matrix* $\mathbf{N}\left(z\right)$ *is divisible (witch zero remainder) by* $d\left(z\right)$.

**Theorem 2**. [4, 6, 8] *The standard matrix (4) is normal if and only if the polynomial* $d(z)$ *is equal to McMillan polynomial* $q(z)$ *of* $\mathbf{T}(z)$.

**3. Structure decomposition of normal transfer matrix**

Let us assume that the polynomial matrix $\mathbf{N}\left(z\right)$ of (4) can be written in the form

(5) $$\mathbf{N}\left(z\right)=\mathbf{P}\left(z\right)\mathbf{Q}\left(z\right)+d\left(z\right)\mathbf{G}\left(z\right)$$

where $\mathbf{P}\left(z\right)\in R^{p}\left[z\right]$, $\mathbf{Q}\left(z\right)\in R^{1\times m}\left[z\right]$, $\mathbf{G}\left(z\right)\in R^{p\times m}\left[z\right]$ and $d\left(z\right)$ is the minimal common denominator of (4).

Substitution (5) into (4) yields

(6) $$\mathbf{T}\left(z\right)=\frac{\mathbf{P}\left(z\right)\mathbf{Q}\left(z\right)}{d\left(z\right)}+\mathbf{G}\left(z\right)$$

**Definition 2.** *It is said that there exists a structure decomposition of* $\mathbf{T}\left(z\right)$ *if and only if it can be written in the form (6).*

**Theorem 3.** *There exists a structure decomposition (6) of (4) if and only if the matrix* $\mathbf{T}\left(z\right)$ *is normal.*

Proof. Necessity. Let $\mathbf{N}_{k,l}^{i,j}\left(z\right)$ be the second order minor composed of the $i$ and $j$ rows and $k$ and $l$ columns of the matrix $\mathbf{N}\left(z\right)$. If (5) holds then

(7) $$\mathbf{N}_{k,l}^{i,j}\left(z\right)=\begin{vmatrix} p_i\left(z\right)q_k\left(z\right)+d\left(z\right)g_{ik}\left(z\right) & p_i\left(z\right)q_l\left(z\right)+d\left(z\right)g_{il}\left(z\right)\\ p_j\left(z\right)q_k\left(z\right)+d\left(z\right)g_{jk}\left(z\right) & p_j\left(z\right)q_l\left(z\right)+d\left(z\right)g_{jl}\left(z\right)\end{vmatrix}=d\left(z\right)n_{kl}^{ij}\left(z\right)\ \text{for}\ \begin{matrix} i,j=1,\ldots,p\\ k,l=1,\ldots,m\end{matrix}$$

where $p_i\left(z\right)$, $q_k\left(z\right)$ and $g_{ik}\left(z\right)$ are the entries of the matrices $\mathbf{P}\left(z\right)$, $\mathbf{Q}\left(z\right)$ and $\mathbf{G}\left(z\right)$, respectively and $n_{kl}^{ij}\left(z\right)$ is a polynomial.

From (7) it follows that the minor $\mathbf{N}_{k,l}^{i,j}\left(z\right)$ is divisible by $d\left(z\right)$. Therefore, the matrix (6) is normal.

Sufficiency. Applying the elementary row and column operations [5, 8] it is possible to reduce thepolynomial matrix $\mathbf{N}\left(z\right)$ to the form

(8) $$\mathbf{U}\left(z\right)\mathbf{N}\left(z\right)\mathbf{V}\left(z\right)=i\left(z\right)\begin{bmatrix} 1 & r\left(z\right)\\ c\left(z\right) & \bar{\mathbf{N}}\left(z\right)\end{bmatrix}$$

where $\mathbf{U}\left(z\right)\in R^{p\times p}\left[z\right]$ and $\mathbf{V}\left(z\right)\in R^{m\times m}\left[z\right]$ are unimodular matrices of elementary row and column operations respectively and $i\left(z\right)\in R\left[z\right]$, $r\left(z\right)\in R^{1\times(m-1)}\left[z\right]$, $c\left(z\right)\in R^{p-1}\left[z\right]$, $\bar{\mathbf{N}}\left(z\right)\in R^{(p-1)\times(m-1)}\left[z\right]$.

If the matrix $\mathbf{T}\left(z\right)$ is normal then every nonzero second order minor of the matrix (8) is divisible by $d\left(z\right)$ and we have

(9) $$i\left(z\right)\left[\bar{\mathbf{N}}\left(z\right)-c\left(z\right)r\left(z\right)\right]=d\left(z\right)\tilde{\mathbf{N}}\left(z\right)$$

for some $\tilde{\mathbf{N}}\left(z\right)\in R^{(p-1)\times(m-1)}\left[z\right]$

Defining

$$\mathbf{P}\left(z\right)=\mathbf{U}^{-1}\left(z\right)\begin{bmatrix}1\\ c\left(z\right)\end{bmatrix},\qquad \mathbf{Q}\left(z\right)=\begin{bmatrix}1 & r\left(z\right)\end{bmatrix}\mathbf{V}^{-1}\left(z\right)$$

(10) $$\mathbf{G}\left(z\right)=\mathbf{U}^{-1}\left(z\right)\begin{bmatrix}0 & 0_{1,m-1}\\ 0_{p-1,1} & d\left(z\right)\tilde{\mathbf{N}}\left(z\right)\end{bmatrix}\mathbf{V}^{-1}\left(z\right)$$

from (8) – (10) we obtain

$$\mathbf{N}(z)=\mathbf{U}^{-1}(z)i(z)\begin{bmatrix}1 & r(z)\\ c(z) & \bar{\mathbf{N}}(z)\end{bmatrix}\mathbf{V}^{-1}(z)=$$
$$=\mathbf{U}^{-1}(z)\left\{i(z)\begin{bmatrix}1\\ c(z)\end{bmatrix}\begin{bmatrix}1 & r(z)\end{bmatrix}+\begin{bmatrix}0 & 0_{1,m-1}\\ 0_{p-1,1} & d(z)\tilde{\mathbf{N}}(z)\end{bmatrix}\right\}\mathbf{V}^{-1}(z)=$$
$$=i(z)\mathbf{P}(z)\mathbf{Q}(z)+d(z)\mathbf{G}(z)$$

Therefore if $\mathbf{T}(z)$ is normal then (6) holds

From proof of sufficiency we have the following procedure for computation of the structure decomposition (6) of the normal matrix $\mathbf{T}(z)$.

**Procedure 1.**

**Step 1**. Using the elementary row and column operations transform the polynomial matrix $\mathbf{N}(z)$ to the m (8) and find the unimodular matrices $\mathbf{U}(z)$, $\mathbf{V}(z)$ and $i(z)$, $r(z)$, $c(z)$ and $\bar{\mathbf{N}}(z)$.

**Step 2.** Using (9) find the polynomial matrix $\tilde{\mathbf{N}}(z)$

**Step 3.** Using (10) find polynomial matrices $\mathbf{P}(z)$, $\mathbf{Q}(z)$ and $\mathbf{G}(z)$.

**Step 4.** Find the desired structure decomposition (6) of $\mathbf{T}(z)$.

Example. Consider the positive system (1) with

(11)
$$\mathbf{A}=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & a\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\quad \mathbf{D}=[0]\quad (a>0)$$

The transfer matrix of the system has the form

(12)
$$\mathbf{T}(z)=\mathbf{C}[\mathbf{I}_n z-\mathbf{A}]^{-1}\mathbf{B}+\mathbf{D}=\frac{\mathbf{N}(z)}{d(z)}$$

where $\mathbf{N}(z)=\begin{bmatrix}z-a & 0\\ 0 & (z-1)^2\end{bmatrix}$, $d(z)=(z-1)^2(z-a)$.

It is easy to check that the transfer matrix (12) is standard and normal if and only if $a\neq 1$ and for $a=1$ is reducible and takes the form

(13)
$$\mathbf{T}(z)=\frac{1}{(z-1)^2}\begin{bmatrix}1 & 0\\ 0 & z-1\end{bmatrix}$$

It is easy to see that the matrix (13) is not normal.

Using the Procedure 1 we shall find the structure decomposition of the transfer matrix (12) for $a=2$. Using the Procedure we obtain successively

Step 1. Using the unimodular matrices

(14)
$$\mathbf{U}(z)=\begin{bmatrix}1 & 1\\ (z-1)^2 & z(z-2)\end{bmatrix},\quad \mathbf{V}(z)=\begin{bmatrix}-z & (z-1)^2\\ 1 & 2-z\end{bmatrix}$$

of the elementary row and column operations we obtain

$$\mathbf{U}(z)\mathbf{N}(z)\mathbf{Z}(z)=\begin{bmatrix}1 & 0\\ 0 & (z-1)^2(z-2)\end{bmatrix}$$

and $i(z)=1$, $r(z)=0$, $c(z)=0$, $\bar{\mathbf{N}}(z)=(z-1)^2(z-2)$.

Step 2. In this case from (9) we have $\tilde{\mathbf{N}}(z)=1$.

Step 3. Using (10) and (14) we obtain

$$\mathbf{P}(z)=\mathbf{U}^{-1}(z)\begin{bmatrix}1\\c(z)\end{bmatrix}=\begin{bmatrix}-z(z-2) & 1\\(z-1)^2 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-z(z-2)\\(z-1)^2\end{bmatrix}$$

$$\mathbf{Q}(z)=\begin{bmatrix}1 & r(z)\end{bmatrix}\mathbf{V}^{-1}(z)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z-2 & (z-1)^2\\1 & z\end{bmatrix}=\begin{bmatrix}z-2 & (z-1)^2\end{bmatrix}$$

$$\mathbf{G}(z)=\mathbf{U}^{-1}(z)\begin{bmatrix}0 & 0_{1,m-1}\\0_{p-1,1} & d(z)\tilde{\mathbf{N}}(z)\end{bmatrix}\mathbf{V}^{-1}(z)=(z-1)^2(z-2)\begin{bmatrix}1 & z\\-1 & -z\end{bmatrix}$$

Step 4. The desired decomposition of (12) has the form

(15)
$$\mathbf{T}(z)=\frac{\mathbf{P}(z)\mathbf{Q}(z)}{d(z)}+\mathbf{G}(z)=\frac{1}{(z-1)^2(z-2)}\begin{bmatrix}-z(z-2)\\(z-1)^2\end{bmatrix}\times$$
$$\begin{bmatrix}z-2 & (z-1)^2\end{bmatrix}+(z-1)^2(z-2)\begin{bmatrix}1 & z\\-1 & -z\end{bmatrix}$$

**4. Realization problem for normal transfer matrices**

**Definition 3**. *Matrices (2) satisfying the equality (3) are called a positive realization of the transfer matrix* $\mathbf{T}(z)$. *The realization (2) is called minimal if and only if the dimension* $n$ *of the matrix* $\mathbf{A}$ *is minimal among all realization of* $\mathbf{T}(z)$.

From (2) it follows that

(16)
$$\mathbf{D}=\lim_{z\to\infty}\mathbf{T}(z)$$

since $\lim_{z\to\infty}\left[\mathbf{I}_n z-\mathbf{A}\right]^{-1}=0$.

The strictly proper transfer function is given by

(17)
$$\mathbf{T}_{sp}(z)=\mathbf{T}(z)-\mathbf{D}=\mathbf{C}\left[\mathbf{I}_n z-\mathbf{A}\right]^{-1}\mathbf{B}$$

Let

(18)
$$d(z)=(z-z_1)^{m_1}(z-z_2)^{m_1}\dots(z-z_p)^{m_p}\,,\quad \sum_{i=1}^{p}m_i=n$$

In this case the strictly proper transfer matrix (17) can be written in the form

(19)
$$\mathbf{T}_{sp}(z)=\sum_{i=1}^{p}\sum_{j=1}^{m_i}\frac{\mathbf{T}_{ij}}{(z-z_i)^{m_i-j+1}}$$

where

(20)
$$\mathbf{T}_{ij}=\frac{1}{(j-1)!}\frac{d^{\,j-1}}{dz^{j-1}}\left[(z-z_i)^{m_i}\,\mathbf{T}_{sp}(z)\right]_{|z=z_i}\qquad \text{for}\quad \begin{matrix}i=1,\dots,p\\ j=1,\dots,m_i\end{matrix}$$

Let us assume that the transfer matrix $\mathbf{T}(z)$ is normal. It is easy to show that the matrix $\mathbf{T}_{sp}(z)$ is also normal. We shall assume that (17) is given in the form

(21)
$$\mathbf{T}_{sp}(z)=\frac{\mathbf{P}(z)\mathbf{Q}(z)}{d(z)}+\mathbf{G}(z)$$

**Theorem 4**. *If the transfer matrix* $\mathbf{T}(z)$ *is normal then the matrices* $\mathbf{T}_{ij}$ *are independent of the matrix* $\mathbf{G}(z)$ *and*

(22)
$$\mathbf{T}_{ij}=\frac{1}{(j-1)!}\frac{d^{\,j-1}}{dz^{j-1}}\left[\frac{\mathbf{P}(z)\mathbf{Q}(z)}{d_i(z)}\right]_{|z=z_i}\qquad \textit{for}\quad \begin{matrix}i=1,\dots,p\\ j=1,\dots,m_i\end{matrix}$$

*where*

(23)
$$d_i(z)=\frac{d(z)}{(z-z_i)^{m_i}}=\prod_{\substack{j=1\\ j\neq i}}^{p}(z-z_j)^{m_j}$$

Proof. From (20), (21) and (23) we have

$$\mathbf{T}_{ij}=\frac{1}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\left\{(z-z_i)^{m_i}\left[\frac{\mathbf{P}(z)\mathbf{Q}(z)}{d(z)}+\mathbf{G}(z)\right]\right\}_{|z=z_i}=\frac{1}{(j-1)!}\frac{d^{j-1}}{dz^{j-1}}\left[\frac{\mathbf{P}(z)\mathbf{Q}(z)}{d_i(z)}\right]_{|z=z_i}$$

since

$$\frac{d^{j-1}}{dz^{j-1}}\left[(z-z_i)^{m_i}\mathbf{G}(z)\right]_{|z=z_i}=\sum_{k=0}^{j-1}\frac{(j-1)!}{k!(j-k-1)!}\frac{d^{j-k-1}}{dz^{j-k-1}}(z-z_i)^{m_i}\frac{d^k}{dz^k}\mathbf{G}(z)_{|z=z_i}=0$$

for $j=1,\dots,m_i$ and $i=1,\dots,p$.

The impulse response matrix $g_i$ is the original of $\mathbf{T}(z)$, i.e.

(24) $$g_i=Z^{-1}\left[\mathbf{T}(z)\right]$$

where $Z^{-1}$ is the inverse $z-$transform operator.

From (24), (19) and (22) we have following important corollary.

**Corollary 1.** *The impulse response matrix $g_i$ of the system (1) with normal transfer matrix is independent of the matrix $\mathbf{G}(z)$.*

**Corollary 2.** *If the transfer matrix $\mathbf{T}(z)$ is normal then*

(25) $$\operatorname{rank}\mathbf{T}_{ij}\le 1;\quad \textit{for}\quad i=1,\dots,p;\ j=1,\dots,m_i$$

The condition (25) follows from formula (22) and the fact that $\mathbf{P}(z)$ is the column polynomial matrix and $\mathbf{Q}(z)$ is the row polynomial matrix.

The positive realization problem for normal matrices $\mathbf{T}(z)$ can be stated as follows.

Given a normal proper rational matrix $\mathbf{T}(z)\in R^{p\times m}(z)$. Find a minimal realization

(26) $$\mathbf{A}_J\in R_+^{n\times n},\ \mathbf{B}\in R_+^{n\times m},\ \mathbf{C}\in R_+^{p\times n},\ \mathbf{D}\in R_+^{p\times m}$$

with the matrix $\mathbf{A}_J$ in the Jordan form

(27a) $$\mathbf{A}_J=\operatorname{diag}\left[\mathbf{J}_1\quad \mathbf{J}_2\quad \dots\quad \mathbf{J}_p\right]$$

where

(27b) $$\mathbf{J}_i=\begin{bmatrix} z_i & 1 & 0 & \dots & 0 & 0\\ 0 & z_i & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & z_i & 1\\ 0 & 0 & 0 & \dots & 0 & z_i\end{bmatrix}\in R^{m_i\times m_i}\quad\text{or}\quad \mathbf{J}_i^{'}=\begin{bmatrix} z_i & 0 & 0 & \dots & 0 & 0\\ 1 & z_i & 0 & \dots & 0 & 0\\ 0 & 1 & z_i & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & z_i\end{bmatrix}\in R^{m_i\times m_i}$$

It is known [3] that the matrix $\mathbf{A}_J$ is cyclic and $\mathbf{T}(z)$ is normal if and only if the $z_i$ with multiplicity $m_i$ corresponds only to one matrix $\mathbf{J}_i$ or $\mathbf{J}_i^{'}$ and $z_1,z_2,\dots,z_p$ are the real roots with multiplicities $m_1,m_2,\dots,m_p$ of $d(z)=0$.

Let

(28a) $$\mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\\ \vdots\\ \mathbf{B}_p\end{bmatrix}\quad\text{and}\quad \mathbf{C}=[\mathbf{C}_1\quad \mathbf{C}_2\quad \dots\quad \mathbf{C}_p]$$

where

(28b) $$\mathbf{B}_i=\begin{bmatrix}\mathbf{B}_{i1}\\ \mathbf{B}_{i2}\\ \vdots\\ \mathbf{B}_{im_i}\end{bmatrix}\quad\text{and}\quad \mathbf{C}_i=[\mathbf{C}_{i1}\quad \mathbf{C}_{i2}\quad \dots\quad \mathbf{C}_{im_i}]\quad\text{for}\quad i=1,\dots,p$$

Taking into account that

(29)
$$\left[\mathbf{I}_{m_i} z - \mathbf{J}_i\right]^{-1} = \begin{bmatrix} \frac{1}{z - z_i} & \frac{1}{(z - z_i)^2} & \dots & \frac{1}{(z - z_i)^{m_i}} \\ 0 & \frac{1}{z - z_i} & \dots & \frac{1}{(z - z_i)^{m_i - 1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{z - z_i} \end{bmatrix} \quad for \quad i = 1, \dots, p$$

we may write

(30)
$$\mathbf{C}_i \left[\mathbf{I}_{m_i} z - \mathbf{J}_i\right]^{-1} \mathbf{B}_i = \frac{1}{z - z_i} \sum_{k=1}^{m_i} \mathbf{C}_{ik} \mathbf{B}_{ik} + \frac{1}{(z - z_i)^2} \sum_{k=1}^{m_i - 1} \mathbf{C}_{ik} \mathbf{B}_{ik+1} + \dots + \frac{1}{(z - z_i)^{m_i}} \mathbf{C}_{ik} \mathbf{B}_{im_i}$$

From comparison of (30) and (19) we have

(31)
$$\mathbf{T}_{ij} = \sum_{k=1}^{j} \mathbf{C}_{ik} \mathbf{B}_{i, m_i - j + k} \quad for \ j = 1, \dots, m_i \ and \ i = 1, \dots, p$$

From (31) for $j = 1$ we obtain

(32)
$$\mathbf{T}_{i1} = \mathbf{C}_{i1} \mathbf{B}_{i, m_i} \quad i = 1, \dots, p$$

Taking into account (25) we decompose $\mathbf{T}_{i1}$ into a column matrix $\mathbf{C}_{i1}$ and row matrix $\mathbf{B}_{i, m_i}$.

From (31) for $j = 2$ we have

(33)
$$\mathbf{T}_{i2} = \mathbf{C}_{i1} \mathbf{B}_{i, m_i - 1} + \mathbf{C}_{i2} \mathbf{B}_{i, m_i} \quad i = 1, \dots, p$$

Knowing $\mathbf{C}_{i1}$ and $\mathbf{B}_{i, m_i}$ we chose as $\mathbf{C}_{i2}$ the column of $\mathbf{T}_{i2}$ that corresponds to the first nonzero entry of $\mathbf{B}_{i, m_i}$ multiplied by the entry and we compute

(34)
$$\mathbf{T}_{i2}^{(1)} = \mathbf{T}_{i2} - \mathbf{C}_{i2} \mathbf{B}_{i, m_i} = \mathbf{C}_{i1} \mathbf{B}_{i, m_i - 1} \quad i = 1, \dots, p$$

Knowing $\mathbf{T}_{i2}^{(1)}$ and $\mathbf{C}_{i1}$ we can find $\mathbf{B}_{i, m_1 - 1}$.

From (31) for $j = 3$ we have

(35)
$$\mathbf{T}_{i3} = \mathbf{C}_{i1} \mathbf{B}_{i, m_i - 2} + \mathbf{C}_{i2} \mathbf{B}_{i, m_i - 1} + \mathbf{C}_{i3} \mathbf{B}_{i, m_i} \quad i = 1, \dots, p$$

Knowing $\mathbf{T}_{i3}$ and $\mathbf{C}_{i2}$, $\mathbf{B}_{i, m_i - 1}$ we may find

(36)
$$\mathbf{T}_{i3}^{(1)} = \mathbf{T}_{i3} - \mathbf{C}_{i2} \mathbf{B}_{i, m_i - 1} = \mathbf{C}_{i1} \mathbf{B}_{i, m_i - 2} + \mathbf{C}_{i3} \mathbf{B}_{i, m_i} \quad i = 1, \dots, p$$

and next in a similar way as for $\mathbf{T}_{i2}$ we may find $\mathbf{C}_{i3}$ and $\mathbf{B}_{i, m_i - 2}$.

Continuing the procedure we may find $\mathbf{C}_{i1}, \mathbf{C}_{i2}, \dots, \mathbf{C}_{i, m_i}$ and $\mathbf{B}_{i1}, \mathbf{B}_{i2}, \dots, \mathbf{B}_{i, m_i}$ for $i = 1, \dots, p$.

In particular case for $m_i = 1$, $i = 1, \dots, p$ when $z_1, z_2, \dots, z_n$ are nonnegative andreal we have

(37)
$$\mathbf{A}_J = \operatorname{diag}\left[z_1, z_2, \dots, z_n\right] \in R_+^{n \times n}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_n \end{bmatrix} \in R_+^{n \times m}, \quad \mathbf{C} = \left[\mathbf{C}_1 \quad \mathbf{C}_2 \quad \dots \quad \mathbf{C}_n\right] \in R_+^{p \times n}$$

and

(38)
$$\mathbf{T}_i = \lim_{z \to z_i} (z - z_i) \mathbf{T}(z) = \mathbf{C}_i \mathbf{B}_i \quad for \ j = 1, \dots, n$$

It is easy to verify that for (27) and (28) the pair $(\mathbf{A}_J, \mathbf{B})$ is controllable and the pair $(\mathbf{A}_J, \mathbf{C})$ is observable. Therefore, the realization $(\mathbf{A}_J, \mathbf{B}, \mathbf{C})$ is minimal.

From (16) it follows that $\mathbf{D} \in R_+^{p \times m}$ if and only if $\mathbf{T}(\infty) \in R_+^{p \times m}$. The matrix $\mathbf{A}_J \in R_+^{n \times n}$ if and only if

$$z_i \geq 0 \quad for \ i = 1, \dots, p$$

If $z_1, z_2, \dots, z_n$ are distinct ($m_i = 1, i = 1, \dots, p = n$) and $\mathbf{T}_k \in R_+^{p \times m}$ then the pair $(\mathbf{A}_J, \mathbf{B})$ is controllable and the pair $(\mathbf{A}_J, \mathbf{C})$ is observable if and only if $\mathbf{T}_k \neq 0$ for $k = 1, \dots, n$. In this case by (25) $\operatorname{rank} \mathbf{T}_k = 1$, $k = 1, \dots, n$ and $\mathbf{T}_k = \mathbf{C}_k \mathbf{B}_k$ with $\operatorname{rank} \mathbf{C}_k = \operatorname{rank} \mathbf{B}_k = 1$, $\mathbf{C}_k \in R_+^{p}$, $\mathbf{B}_k \in R_+^{1 \times m}$ for $k = 1, \dots, n$.

Therefore the following theorem has been proved.

**Theorem 5**. *There exists a positive minimal realization (26) of the normal proper matrix (4) with distinct real roots* $z_1, z_2, \dots, z_n$ *of* $d(z)=0$ *if and only if the following conditions are satisfied:*

(39) $$\mathbf{T}(\infty) \in R_+^{p\times m}$$

(40) $$z_k \geq 0 \quad \text{for } k=1,\dots,n$$

(41) $$\mathbf{T}_k \in R_+^{p\times m}, \quad \text{for } k=1,\dots,n$$

**Remark 1.** *When* $m_i > 1$ *for at least one* $i$ *then the condition (41) should be modified by assuming that the matrices* $\mathbf{T}_{ij}$ *can be decomposed according to (31) so that* $\mathbf{B} \in R_+^{n\times m}$, $\mathbf{C} \in R_+^{p\times n}$ *and the pair* $(\mathbf{A}_J, \mathbf{B})$ *is controllable and the pair* $(\mathbf{A}_J, \mathbf{C})$ *is observable.*

A positive realization (26) of the normal proper matrix (4) can be computed by the use of the following procedure.

**Procedure 2.**

**Step 1.** Using (16) and (17) compute the matrix $\mathbf{D}$ and the strictly proper matrix $\mathbf{T}_{sp}(z)$.

**Step 2.** Compute the roots $z_1, z_2, \dots, z_p$ and their multiplicities $m_1, m_2, \dots, m_i$ of the equation $d(z)=0$.

**Step 3.** Using the formula (20) or (22) compute the matrices $\mathbf{T}_{ij}$ for $j=1,\dots,m_i$, $i=1,\dots,p$

**Step 4.** Using (31) find the columns $\mathbf{C}_{ij}$ of $\mathbf{C}$ and the rows $\mathbf{B}_{ij}$ of $\mathbf{B}$ for $i=1,\dots,p$, $j=1,\dots,m_i$.

**Step 5.** Find the desired realization (26) using (27) and (28).

**Example 2**. Compute the positive realization (26) of the transfer matrix

(42) $$\mathbf{T}(z) = \frac{1}{(z-1)^2(z-2)}\begin{bmatrix} z-2 & 0 \\ 0 & (z-1)^2 \end{bmatrix}$$

It is easy to check that the matrix (42) is normal. Using Procedure 2 we obtain successively.

Step 1. The matrix $\mathbf{D}=0$ since (42) is strictly proper and $\mathbf{T}_{sp}(z)=\mathbf{T}(z)$.

Step 2. In this case $d(z)=(z-1)^2(z-2)$ and $z_1=1$, $m_1=2$, $z_2=2$, $m_2=1$.

Step 3. Using the formula (20) we obtain

$$\mathbf{T}_{11} = \lim_{z\to z_1}(z-z_1)^{m_1}\mathbf{T}(z) = \frac{1}{z-2}\begin{bmatrix} z-2 & 0 \\ 0 & (z-1)^2 \end{bmatrix}_{|z=1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(43) $$\mathbf{T}_{12} = \frac{d}{dz}\left[(z-z_1)^{m_1}\mathbf{T}(z)\right]_{|z=z_1} = \frac{d}{dz}\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{(z-1)^2}{z-2} \end{bmatrix}_{|z=1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{T}_{21} = \lim_{z\to z_2}(z-z_2)^{m_2}\mathbf{T}(z) = \frac{1}{(z-1)^2}\begin{bmatrix} z-2 & 0 \\ 0 & (z-1)^2 \end{bmatrix}_{|z=2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that the same result can be obtained by the use of the formula (22)

Step 4. Using (31) and (43) we obtain

$$\mathbf{T}_{11} = \mathbf{C}_{11}\mathbf{B}_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{C}_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{B}_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathbf{T}_{12} = \mathbf{C}_{11}\mathbf{B}_{11} + \mathbf{C}_{12}\mathbf{B}_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{B}_{11} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \mathbf{C}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{T}_{21} = \mathbf{C}_{21}\mathbf{B}_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \mathbf{C}_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{B}_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$