Chapter 5

Orthogonal Filter Banks

5.1 Paraunitary Matrices

For an orthogonal matrix, the inverse is the transpose. When the matrix is 2 by 2, this imposes a tremendous constraint. Suppose we choose just one entry of the matrix, in the upper left corner. Our choice should not exceed 1 in absolute value, and we call it $\cos\theta$:

$$H = \begin{bmatrix} \cos\theta & - \\ - & - \end{bmatrix}.$$

The other entries are almost completely determined by this choice. Below $\cos\theta$ we need $\sin\theta$ or $-\sin\theta$, to make the first column a unit vector. The second column must be a unit vector orthogonal to the first one, and we select

$$H = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{5.1}$$

$\boldsymbol{H}$ rotates every plane vector $\boldsymbol{x}$ by θ. *The length is preserved*: $\|\boldsymbol{Hx}\| = \|\boldsymbol{x}\|$.

This is not the only orthogonal matrix that starts with $\cos\theta$. We could multiply the second column or the second row by -1. Then the rotation matrix in (5.1) becomes a *reflection* matrix; its determinant changes to -1. If these are *complex* matrices, we could multiply any column and any row by numbers on the unit circle $|z| = 1$. This yields every unitary 2 by 2 matrix. Essentially, one entry determines the whole matrix. The inverse is the conjugate transpose: $\boldsymbol{H}^{-1} = \overline{\boldsymbol{H}}^T$.

An analysis bank is represented by an infinite matrix (in the time domain). But in the frequency domain or z-domain, the matrix is 2 by 2 (from $M = 2$ channels and $M = 2$ phases). This matrix depends on a parameter ω or z. Therefore we stretch the definition from unitary to ***paraunitary***:

Definition 5.1 *The matrix $\boldsymbol{H}(z)$ is paraunitary if it is unitary for all $|z| = 1$:*

$$\boldsymbol{H}^T(e^{-j\omega})\boldsymbol{H}(e^{j\omega}) = \boldsymbol{I} \quad \textit{for all } \omega. \tag{5.2}$$

This extends to all $z \neq 0$ by $\widetilde{\boldsymbol{H}}(z) = \boldsymbol{H}^T(z^{-1})$. Then a paraunitary matrix has

$$\boldsymbol{H}^T(z^{-1})\boldsymbol{H}(z) = \widetilde{\boldsymbol{H}}(z)\,\boldsymbol{H}(z) = \boldsymbol{I} \quad \textit{for all } z. \tag{5.3}$$

When the coefficients $\boldsymbol{h}(k)$ are complex, they are conjugated in $\widetilde{\boldsymbol{H}}(z)$.

The matrix $\boldsymbol{H}$ need not be 2 by 2. If it is 1 by 1, then $|H(e^{j\omega})| = 1$. The corresponding filter is *allpass*. The best allpass examples are ratios of polynomials coming from IIR filters — since only trivial polynomials z^{-l} can have $|H(e^{j\omega})| = 1$.

If $\boldsymbol{H}(z)$ is $M \times M$, it could come from an M-channel filter bank. It might be the polyphase matrix $\boldsymbol{H}_p(z)$ or the modulation matrix $\boldsymbol{H}_m(z)$ (divided by $\sqrt{2}$). We will show that the filter bank is orthogonal if these matrices are paraunitary. That is the important connection for this book.

Equation (5.2) gives the inverse matrix by transposing and conjugating the original. The synthesis bank comes by "reversing" the analysis bank. Note that for a square matrix, $\boldsymbol{H}(z)$ is paraunitary when $\boldsymbol{H}^{-1}(z)$ and $\boldsymbol{H}^T(z)$ and $\widetilde{\boldsymbol{H}}(z)$ are paraunitary. And notice especially what equation (5.3) says about the *determinants* of these matrices:

$$\left(\det \widetilde{\boldsymbol{H}}(z)\right)\left(\det \boldsymbol{H}(z)\right) = 1. \tag{5.4}$$

The determinants are 1 by 1 allpass!

Theorem 5.1 *If a square paraunitary matrix* $\boldsymbol{H}(z)$ *is FIR (= polynomial), then its determinant must be a delay:*

$$\det \boldsymbol{H}(z) = \pm z^{-l}. \tag{5.5}$$

The determinant of $\boldsymbol{H}_p(z)$ is also a delay for any *bi*-orthogonal filter bank. Orthogonality requires more; the polyphase matrix $\boldsymbol{H}_p(z)$ must be paraunitary.

If $\boldsymbol{H}(z)$ is rectangular, say M by r, then $\widetilde{\boldsymbol{H}}(z)\boldsymbol{H}(z) = \boldsymbol{I}$ is still possible. The identity matrix is now r by r (and necessarily $r \le M$, since the rank cannot exceed M). The matrix is still called paraunitary. There is no inverse matrix $\boldsymbol{H}^{-1}(z)$ in the rectangular case, but $\widetilde{\boldsymbol{H}}(z)$ is a left-inverse. We hope and expect that $\boldsymbol{H}(z)$ can be completed to a square paraunitary matrix, by adding $M-r$ more columns. In the applications, we are starting with r filters and creating $M-r$ additional filters — while preserving orthogonality.

When $\widetilde{\boldsymbol{H}}(z)\boldsymbol{H}(z) = d\boldsymbol{I}$ with $d > 0$, we could still use the word paraunitary. The chief example is the modulation matrix, which has $d = 2$. Some authors keep this flexibility.

Our chief interest is in the case $M = 2$. A 2 by 2 paraunitary matrix is essentially determined by one entry. For a paraunitary polyphase matrix, the even phase $H_{00}(z)$ of the lowpass filter essentially determines the whole orthogonal filter bank. For filter banks with four taps, there are two free parameters — which can be $c(0)$ and $c(2)$. The design problem is greatly reduced when the filter bank is orthogonal and its polyphase matrix is paraunitary.

2 by 2 Paraunitary Examples

The first example is $\boldsymbol{H}(z) = \boldsymbol{I}$. It corresponds to a "lazy filter bank" without filters. The polyphase matrix is $\boldsymbol{H}_p = \boldsymbol{I}$ when the filters are $H_0(z) = 1$ and $H_1(z) = z^{-1}$. The bank just splits the input vector $\boldsymbol{x}$ into odd and even phases, without filtering it.

The next paraunitary matrix is $\boldsymbol{H}(z) = \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. This is still constant. It is the polyphase matrix of the Haar average-difference filter bank. We move on to more interesting examples.

Suppose $\boldsymbol{R}_0, \ldots, \boldsymbol{R}_l$ are constant rotation matrices as in equation (5.1). The rotation angles are $\theta_0, \ldots, \theta_l$. If we multiply those matrices, we get a single rotation through the total angle $\sum \theta_i$. But if we introduce a diagonal matrix $\Lambda(z) = \operatorname{diag}\left(1, z^{-1}\right)$ *between* those rotations, we get something much more general and important:

$$\boldsymbol{H}(z) = \boldsymbol{R}_l\ \Lambda(z)\ \boldsymbol{R}_{l-1}\ \Lambda(z) \cdots \boldsymbol{R}_1\ \Lambda(z)\ \boldsymbol{R}_0. \tag{5.6}$$

This matrix is paraunitary, because it is the product of paraunitary factors. Its determinant is z^{-l} (the product of determinants). The filter bank can be realized as a lattice structure involving l delays. Section 4.5 proved that *this is the most general 2 by 2 paraunitary matrix of degree l.* The only subtle point is the meaning of the word "degree." Here are two small examples to make that point, both with $R_0 = R_2 = I$. Then $H(z)$ is $\Lambda(z)R_1\Lambda(z)$. The only difference is in R_1:

$$\begin{aligned} \boldsymbol{H}(z) &= \begin{bmatrix} 1 & \\ & z^{-1} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & z^{-1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z^{-1} \\ -z^{-1} & z^{-2} \end{bmatrix} \\ \boldsymbol{H}(z) &= \begin{bmatrix} 1 & \\ & z^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \\ & z^{-1} \end{bmatrix} = \begin{bmatrix} 0 & z^{-1} \\ -z^{-1} & 0 \end{bmatrix} \end{aligned}$$

Both have determinant z^{-2}. Both have degree $l = 2$! But the power z^{-2}, which appears in the first, does not appear in the second. The correct definition of degree is the *Smith-McMillan degree — the minimum number of delays* required to realize the system. For paraunitary systems, but not for all systems, the degree is revealed by the determinant.

For lowpass-highpass filters with four taps, the even and odd phases have two terms each. If $H(z)$ is paraunitary, it fits the general form (5.6) with $l = 1$. There are two parameters θ_0 and θ_1 to be chosen. Again we have two design parameters, for 4-tap orthogonal filters, but this time they are angles.

This chapter constructs a family of *maxflat filters*. Then Chapter 6 shows that these filters give the *Daubechies wavelets*. We mention the connection now, so you will attach importance to maxflat filters. They don't have a sharp transition band — they don't have the quickest transition from passband to stopband — but they produce the most vanishing moments.

Problem Set 5.1

1. Suppose $H(z) = h(0) + h(1)z^{-1} + \cdots + h(N)z^{-N}$ is paraunitary with $N > 0$. Show that $h(0)^T h(N) =$ zero matrix. Deduce that $h(0)$ and $h(N)$ are both singular matrices.
2. Multiply out $H(z) = R_1\Lambda(z)R_0$ for angles θ_1 and θ_0. If this is a polyphase matrix, what filter coefficients $d(k)$ come from the even and odd phases in the lower row of $H(z)$? If that is a highpass filter, with $D = 0$ at $\omega = 0$ (which is $z = 1$), show that $\theta_1 + \theta_0 = -\frac{\pi}{4} + n\pi$. (Haar has $\theta_0 = -\frac{\pi}{4}$ and $\theta_1 = 0$.)
3. Complete this polynomial matrix to have $\det H(z) \equiv 1$. Is it paraunitary?
$$H(z) = \begin{bmatrix} 1 + z^{-1} & z^{-1} \\ — & — \end{bmatrix}.$$
4. When can a row be the first row of a paraunitary matrix?
5. Show that $H = I - 2vv^T$ is a unitary matrix, where $v^T v = 1$. Compute H^{-1} and the determinant of H. *Construct an example.*
6. Show that $H = I - vv^T + z^{-1}vv^T$ is a paraunitary matrix for a unit-norm vector v. Compute its inverse and determinant. The factorization of $H_p(z)$ using these Householder matrices is in [V].

5.2 Orthonormal Filter Banks

This section brings together the requirements for an orthonormal filter bank. We will see those requirements in the *time domain* and the *polyphase domain* and the *modulation domain*. These requirements are conditions on the filter coefficients $c(k)$ and $d(k)$. Then equation (5.19) indicates a simple choice of the d's coming from the c's. If the lowpass filter meets the orthogonality requirements, it is easy to construct a highpass filter to go with it.

The discussion is in terms of a 2-channel FIR filter bank, $M = 2$. But the conditions extend immediately to any M. *The polyphase matrix and the modulation matrix must be paraunitary.* In the M-channel case, the lowpass filter does not immediately determine the $M - 1$ remaining filters (which are bandpass). There is some freedom in their construction, and we come back in Chapter 9 to $M > 2$.

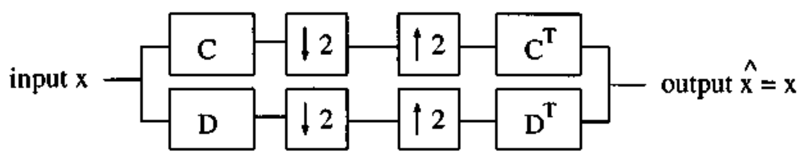


Figure 5.1: An orthogonal filter bank has synthesis bank = transpose of analysis bank.

Figure 5.1 shows the structure of an orthogonal filter bank. We intend to achieve $\widehat{x} = x$, with synthesis filters C^T and D^T that are time-reversals of the analysis filters:

$$\widetilde{C} = C^T \quad \text{and} \quad \widetilde{c}(n) = c(-n) \tag{5.7}$$

$$\widetilde{D} = D^T \quad \text{and} \quad \widetilde{d}(n) = d(-n) \tag{5.8}$$

As it stands, $\widetilde{C}$ and $\widetilde{D}$ are anticausal. At the end we make them causal by N delays. The output $\widehat{x}(n)$ is equally delayed; it is $x(n - N)$. But the algebra is easiest for C^T and D^T with no delays.

This special structure imposes special conditions on the c's and d's for perfect reconstruction. We will call the requirements ***Condition O*** (for orthogonality). This section finds four equivalent forms: Condition O on the infinite matrix, on the lowpass coefficients, and on the polyphase and modulation matrices H_p and H_m. We look first at the infinite matrices in the time domain.

Time Domain: Condition O and the Alternating Flip

The key matrix in the time domain is H_t. It represents the direct form of the analysis bank, with downsampling. The lowpass part $L = (\downarrow 2)C$ comes above the highpass part $B = (\downarrow 2)D$. We display this infinite matrix for filters of length four:

$$H_t = \begin{bmatrix} L \\ B \end{bmatrix} = \begin{bmatrix} c(3) & c(2) & c(1) & c(0) & & \\ & & c(3) & c(2) & c(1) & c(0) \\ & & & & \cdot & \cdot \\ d(3) & d(2) & d(1) & d(0) & & \\ & & d(3) & d(2) & d(1) & d(0) \\ & & & & \cdot & \cdot \end{bmatrix}. \tag{5.9}$$

The shifts by 2 were created by downsampling, which removed the odd-numbered rows.

With orthogonality, the synthesis filters are to be time-reversals of the analysis filters. The infinite synthesis matrix contains the transposes of $(\downarrow 2)C$ and $(\downarrow 2)D$. Those transposes are $C^T(\uparrow 2)$ and $D^T(\uparrow 2)$, since upsampling is the transpose of downsampling:

$$H_t^T = \begin{bmatrix} L^T & B^T \end{bmatrix} = \begin{bmatrix} c(3) & & d(3) & & \\ c(2) & & d(2) & & \\ c(1) & c(3) & d(1) & d(3) & \\ c(0) & c(2) & d(0) & d(2) & \\ & c(1) \quad \cdot & & d(1) \quad \cdot & \\ & c(0) \quad \cdot & & d(0) \quad \cdot & \end{bmatrix}. \tag{5.10}$$

The shifts by 2 in the columns are created by upsampling, which removes every other column. We require $\widehat{x} = x$. This means that $H_t^T H_t = I$. The matrix H_t is required to be an *orthogonal matrix*. Its columns are orthonormal and so are its rows: $H_t H_t^T = I$. We can express this Condition O in matrix form, and in block form, and in coefficient form.

Condition O An orthogonal filter bank comes from an orthogonal matrix:

$$H_t^T H_t = I \text{ and } H_t H_t^T = I. \tag{5.11}$$

In block form this means that

$$\begin{bmatrix} L^T & B^T \end{bmatrix} \begin{bmatrix} L \\ B \end{bmatrix} = L^T L + B^T B = I \tag{5.12}$$

and

$$\begin{bmatrix} L \\ B \end{bmatrix} \begin{bmatrix} L^T & B^T \end{bmatrix} = \begin{bmatrix} LL^T & LB^T \\ BL^T & BB^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5.13}$$

For the coefficients $c(k)$ and $d(k)$, equation (5.13) becomes *orthogonality to double shifts*:

$$LL^T = I: \quad \sum c(n)c(n-2k) = \delta(k) \tag{5.14}$$

$$LB^T = 0: \quad \sum c(n)d(n-2k) = 0 \tag{5.15}$$

$$BB^T = I: \quad \sum d(n)d(n-2k) = \delta(k). \tag{5.16}$$

Because of (5.14)–(5.16), we refer to Condition O as ***double-shift orthogonality***. Its equivalent in the frequency domain is presented below. This double-shift orthogonality immediately rules out odd length filters! If the length is $N+1=5$, a shift by 4 gives an inner product that cannot be zero:

$$(c(0), c(1), c(2), c(3), c(4)) \cdot (0, 0, 0, 0, c(0)) = c(0)c(4) \neq 0.$$

N cannot be even (the filter length cannot be odd) because $\frac{N}{2}$ double shifts would give a shift by N — and the inner product $c(0)c(N)$ is not zero. So N is odd. The degree $2N$ of the halfband product filter P is $2, 6, 10, \ldots$ The clearest examples have $N=3$ and $2N=6$.

Example. Condition O in (5.14) imposes two constraints on four coefficients:

$$c(0)^2 + c(1)^2 + c(2)^2 + c(3)^2 = 1 \quad \text{and} \quad c(0)c(2) + c(1)c(3) = 0. \tag{5.17}$$

Equation (5.16) is an identical condition on the d's, from $BB^T = I$. Equation (5.15) is the orthogonality of the rows of L to the rows of B. Those are the fundamental design constraints on an orthogonal filter bank.

The conditions on the $\boldsymbol{c}$'s and $\boldsymbol{d}$'s are independent, but something good happens. *If the $\boldsymbol{c}$'s satisfy equation* (5.14), *it is easy to choose the $\boldsymbol{d}$'s.* We display a choice of $\boldsymbol{d}$'s that automatically gives orthogonality:

$$\begin{bmatrix} L \\ B \end{bmatrix} = \begin{bmatrix} c(3) & c(2) & c(1) & c(0) & & \\ & & c(3) & c(2) & c(1) & c(0) \\ & & & & \cdot & \cdot \\ -c(0) & c(1) & -c(2) & c(3) & & \\ & & -c(0) & c(1) & -c(2) & c(3) \\ & & & & \cdot & \cdot \end{bmatrix}. \tag{5.18}$$

This is the ***alternating flip***. The $\boldsymbol{c}$'s are reversed in order and alternated in sign, to produce the $\boldsymbol{d}$'s. We start with lowpass coefficients $c(0), \ldots, c(N)$, where N is odd. The highpass coefficients are

$$d(k) = (-1)^k c(N-k). \tag{5.19}$$

The essential point can be checked by eye, in the infinite matrix (5.18). *If the **top** rows are orthogonal to each other, then **all** rows are orthogonal.* The zero dot products in $\boldsymbol{L}\boldsymbol{B}^T$ are:

$$\begin{aligned} -c(3)c(0) + c(2)c(1) - c(1)c(2) + c(0)c(3) &= 0 \\ -c(1)c(0) + c(0)c(1) &= 0. \end{aligned} \tag{5.20}$$

Furthermore, the $\boldsymbol{d}$'s are orthonormal within themselves ($\boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{I}$) because the $\boldsymbol{c}$'s are. Equations (5.17) hold for the $\boldsymbol{d}$'s, when they are constructed by flipping the $\boldsymbol{c}$'s. Minus signs cancel in $d(3)d(1)$ which is $(-c(0))(-c(2))$.

Our four-tap example has $N = 3$. The alternating flip gives $\boldsymbol{L}\boldsymbol{B}^T = 0$ for every odd N. The top rows of $\boldsymbol{H}_t$ in (5.18) are *always* orthogonal to the bottom rows. Also (5.16) for the $\boldsymbol{d}$'s follows from (5.14) for the $\boldsymbol{c}$'s. Thus the alternating flip reduces orthogonality to (5.14):

Condition O on the coefficients: $\sum c(n)c(n-2k) = \delta(k)$.

Polyphase Domain: Condition O and the Alternating Flip

The polyphase form separates $(\downarrow 2)\boldsymbol{C}$ and $(\downarrow 2)\boldsymbol{D}$ into even phase and odd phase. In the time domain, we are rearranging the columns of the matrix $\boldsymbol{H}_t$. All even columns come before the odd columns. The matrix goes into the 2 by 2 block form of Section 4.4, with time-invariant filters as the blocks:

$$\boldsymbol{H}_{\text{block}} = \begin{bmatrix} C_{\text{even}} & C_{\text{odd}} \\ D_{\text{even}} & D_{\text{odd}} \end{bmatrix}.$$

In the z-domain, we are rearranging the response functions $C(z)$ and $D(z)$:

$$\sum c(k)z^{-k} = C_{\text{even}}(z^{-2}) + z^{-1}C_{\text{odd}}(z^{-2}). \tag{5.21}$$

Those phase responses are written $H_{00}(z)$ and $H_{01}(z)$ when $C(z)$ is $H_0(z)$. The highpass response $D(z) = H_1(z)$ decomposes in the same way. *The polyphase matrix is*

$$\boldsymbol{H}_p(z) = \begin{bmatrix} C_{\text{even}}(z) & C_{\text{odd}}(z) \\ D_{\text{even}}(z) & D_{\text{odd}}(z) \end{bmatrix} = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{bmatrix}. \tag{5.22}$$

Now Condition O translates directly into a requirement on $H_p(z)$. A filter bank is **orthogonal** when its polyphase matrix is **paraunitary**:

$$H_p^T(e^{-j\omega})H_p(e^{j\omega}) = I \text{ for all } \omega \text{ and } \widetilde{H}_p(z)H_p(z) = I \text{ for all } z. \tag{5.23}$$

The inverse of $H_p(z)$ is the synthesis polyphase matrix. The matrix and its inverse can be multiplied in either order — here is analysis times synthesis:

$$\begin{bmatrix} C_{\text{even}}(z) & C_{\text{odd}}(z) \\ D_{\text{even}}(z) & D_{\text{odd}}(z) \end{bmatrix} \begin{bmatrix} C_{\text{even}}(z^{-1}) & D_{\text{even}}(z^{-1}) \\ C_{\text{odd}}(z^{-1}) & D_{\text{odd}}(z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.24}$$

On the unit circle, where z^{-1} is $\bar{z}$, row 1 times column 1 becomes

$$|C_{\text{even}}(z)|^2 + |C_{\text{odd}}(z)|^2 = 1 \text{ when } |z| = 1. \tag{5.25}$$

This is the essence of Condition O in the polyphase domain.

For the example with four coefficients, it is helpful to multiply out equation (5.25):

$$\left(c(0) + c(2)z^{-1}\right)(c(0) + c(2)z) + \left(c(1) + c(3)z^{-1}\right)(c(1) + c(3)z) =$$
$$c(0)^2 + c(2)^2 + c(1)^2 + c(3)^2 + [c(0)c(2) + c(1)c(3)]\left(z^{-1} + z\right) = 1.$$

Thus (5.25) is equivalent to the explicit statement (5.17). The sum of squares is 1 and the dot product $c(0)c(2) + c(1)c(3)$ is zero.

The other multiplications in (5.24) give answers 1 or 0 in the same way. All these requirements on the d's are automatically satisfied by the flip construction! We write that choice $d(k) = (-1)^k c(N-k)$ in the z-domain:

$$\sum d(k)z^{-k} = \sum c(N-k)(-z)^{-k} = \sum c(n)(-z)^{n-N}.$$

This relation between highpass and lowpass is an *alternating flip*:

$$D(z) = (-z)^{-N}C(-z^{-1}). \tag{5.26}$$

The number N is odd. Because of $(-z)^{-N}$, the even and odd phases in C are reversed to odd and even phases in D. We take $N = 3$ as typical. An alternating flip of $c(0), c(1), c(2), c(3)$ yields

$$\begin{aligned} D(z) &= c(3) - c(2)z^{-1} + c(1)z^{-2} - c(0)z^{-3} \\ &= \left(c(3) + c(1)z^{-2}\right) - z^{-1}\left(c(2) + c(0)z^{-2}\right). \end{aligned} \tag{5.27}$$

With this flip in row 2, the multiplication $H_p(z)H_p^T(z^{-1})$ becomes

$$\begin{bmatrix} c(0) + c(2)z^{-1} & c(1) + c(3)z^{-1} \\ c(3) + c(1)z^{-1} & -c(2) - c(0)z^{-1} \end{bmatrix} \begin{bmatrix} c(0) + c(2)z & c(3) + c(1)z \\ c(1) + c(3)z & -c(2) - c(0)z \end{bmatrix} = I.$$

The off-diagonal entries of the product are *automatically* zero. *The alternating flip achieves* $LB^T = 0$ *with or without orthogonality*. The 2, 2 entry of the product is the same as the 1, 1 entry, when $|z| = 1$. The orthogonality requirement (5.25) makes the 1, 1 entry equal to 1.

Modulation Domain: Condition O and the Alternating Flip

The function that arises from modulating $C(z)$ is $C(-z)$. The frequency in $z = e^{j\omega}$ changes by π to produce $-z = e^{j(\omega+\pi)}$. This modulation takes $C(\omega)$ to $C(\omega+\pi)$. The frequency response graph is shifted by π.

Our goal is to relate $C(z)$ to $C(-z)$ and $D(z)$ to $D(-z)$ for an orthogonal filter bank. We already know Condition O on the coefficients:

$$c(0)^2 + c(1)^2 + c(2)^2 + c(3)^2 = 1 \quad \text{and} \quad c(0)c(2) + c(1)c(3) = 0.$$

Watch how 1 and 0 appear in $|C(z)|^2$. Stay on the unit circle where $\bar{z}^{-1} = z$:

$$\left(c(0) + c(1)z^{-1} + c(2)z^{-2} + c(3)z^{-3}\right)\left(c(0) + c(1)z + c(2)z^2 + c(3)z^3\right) =$$
$$1 + [c(0)c(1) + c(1)c(2) + c(2)c(3)]\left(z^{-1} + z\right) + c(0)c(3)\left(z^{-3} + z^3\right).$$

Now change z to $-z$. The odd powers z and z^3 change sign. When we add, those odd powers cancel:

$$|C(z)|^2 + |C(-z)|^2 = |C(\omega)|^2 + |C(\omega+\pi)|^2 = 2 \quad \text{for all } z = e^{j\omega}. \tag{5.28}$$

This is the *halfband condition*, also called the *Nyquist condition*: $|C(z)|^2$ is a halfband filter. Those filters we can design! In other words, the lowpass analysis filter $C(z)$ is a spectral factor of a halfband filter.

Condition O on $H_m(z)$. *The modulation matrix of an orthogonal filter bank is a paraunitary matrix times* $\sqrt{2}$:

$$H_m(z)\widetilde{H}_m(z) = 2I \quad \text{for all } z. \tag{5.29}$$

On the circle $z = e^{j\omega}$, the modulation matrix is a unitary matrix times $\sqrt{2}$:

$$\begin{bmatrix} C(\omega) & C(\omega+\pi) \\ D(\omega) & D(\omega+\pi) \end{bmatrix}\begin{bmatrix} \overline{C(\omega)} & \overline{D(\omega)} \\ \overline{C(\omega+\pi)} & \overline{D(\omega+\pi)} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{5.30}$$

The 1, 1 entry of this matrix product is $|C(\omega)|^2 + |C(\omega+\pi)|^2 = 2$ by (5.28). The other entries, when we multiply them out, follow immediately from (5.15) and (5.16). Thus Condition O on the coefficients is equivalent to Condition O on the modulation matrix H_m. It is also equivalent to Condition O on H_p. It is the statement that the analysis bank followed by its transpose gives perfect reconstruction. We summarize:

Theorem 5.2 *For an orthogonal filter bank the lowpass coefficients must satisfy* Condition O *(we give four equivalent forms).*

Matrix form	$LL^T = (\downarrow 2)CC^T(\uparrow 2) = I$
Coefficient form	$\sum c(n)c(n-2k) = \delta(k)$
Polyphase form	$\lvert C_{\text{even}}(e^{j\omega})\rvert^2 + \lvert C_{\text{odd}}(e^{j\omega})\rvert^2 = 1$
Modulation form	$\lvert C(\omega)\rvert^2 + \lvert C(\omega+\pi)\rvert^2 = 2$

By an alternating flip, $LB^T = 0$ and $BB^T = I$ follow immediately from the lowpass part $LL^T = I$. The real problem is the design of the lowpass filter.

Symmetry Prevents Orthogonality

It is natural to want two good properties at once. Symmetry is good for the eye, and orthogonality is good for the algorithm. But the only filters with both properties are averaging filters (Haar filters) with two coefficients. We are forced to use IIR filters, or M channels, or multifilters with matrix coefficients (Section 7.5). Extra computation is unavoidable, because the next theorem rules out a perfect filter.

Theorem 5.3 *A symmetric orthogonal FIR filter can only have two nonzero coefficients.*

Proof. N is odd for orthogonality. The filter length must be even. With $N = 5$ a symmetric filter of length 6 has the form $(c(0), c(1), c(2), c(2), c(1), c(0))$. This vector must be orthogonal to all its double shifts. The inner product with its shift by *four* must be $2c(0)c(1) = 0$. Therefore $c(1) = 0$. Then the inner product with its shift by *two* gives $2\,c(0)c(2) = 0$. The only nonzero coefficient is $c(0)$ at both ends of the filter. This completes the proof.

By convention $c(0)$ is the first nonzero coefficient. Shift the filter if necessary to achieve this. The only symmetric orthogonal possibilities are $c = (1, 1)/\sqrt{2}$ and $(1, 0, 0, 1)/\sqrt{2}$ and $(1, 0, \ldots, 0, 1)/\sqrt{2}$. Only the Haar coefficients $(1, 1)/\sqrt{2}$ will lead to orthogonal wavelets. Symmetry really conflicts with orthogonality.

A second proof observes that the odd phase is the flip of the even phase:

$$(c(0), c(4), c(2), c(2), c(4), c(0)) \text{ has } |C_{\text{even}}(z)|^2 = |C_{\text{odd}}(z)|^2.$$

Condition O is $|C_{\text{even}}(z)|^2 + |C_{\text{odd}}(z)|^2 = 2$. With symmetry this separates into $|C_{\text{even}}(z)|^2 = 1$ and $|C_{\text{odd}}(z)|^2 = 1$. *The even phase is an allpass filter* ! So is the odd phase. But FIR allpass filters can only have one nonzero coefficient, which completes the second proof.

A third proof is based on the zeros of $C(z)$. This is in Section 5.4 below.

Problem Set 5.2

1. (a) Show that the alternating flip with odd N gives $\overline{D(\omega)} = -e^{iN\omega}C(\omega + \pi)$.
   (b) Then $\overline{D(\omega+\pi)} = e^{iN\omega}C(\omega)$. Verify that $C(\omega)\overline{D(\omega)} + C(\omega+\pi)\overline{D(\omega+\pi)} = 0$.
   (c) Also $|C(\omega)|^2 + |C(\omega+\pi)|^2 = 2$ implies that $|D(\omega)|^2 + |D(\omega+\pi)|^2 = 2$
2. For any four coefficients $h(0), \ldots, h(3)$, verify that
$$|H_{\text{even}}(z)|^2 + |H_{\text{odd}}(z)|^2 = \tfrac{1}{2}\left(|H(z)|^2 + |H(-z)|^2\right).$$
   Then Condition O for polyphase equals Condition O for modulation.
3. Find the flaw in this construction of the modulation matrix $H_m(z)$. Start with an arbitrary upper left entry $C(z)$. Complete the 2 by 2 matrix to be paraunitary (times $\sqrt{2}$). Then the filter bank is orthogonal.
4. Find d by an alternating flip of $c = (c(0), \ldots, c(5))$. Verify equation (5.15) directly to show that c is double-shift orthogonal to d.
5. Verify that $c = \frac{1}{4\sqrt{2}}(1+\sqrt{3}, 3+\sqrt{3}, 3-\sqrt{3}, 1-\sqrt{3})$ satisfies Condition O (Daubechies).
6. If two lowpass filters C and H satisfy Condition O, does their product satisfy Condition O?
7. If two polyphase matrices $H_p(z)$ and $K_p(z)$ satisfy Condition O (they are paraunitary), does their product satisfy Condition O?

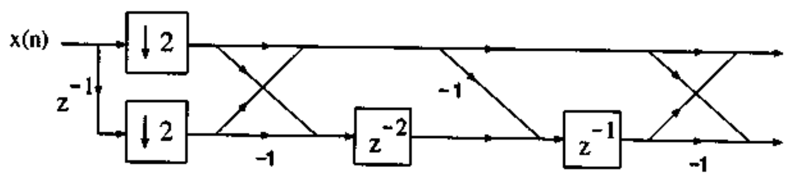
8. If two modulation matrices $\boldsymbol{H}_m(z)$ and $\boldsymbol{K}_m(z)$ satisfy Condition O, show that $\boldsymbol{H}_m(z)\boldsymbol{K}_m(z)/\sqrt{2}$ is also paraunitary. What is the lowpass filter in the product?

9. Why does orthogonality *require* an alternating flip between the lowpass filter C and the highpass filter D? Explain why the paraunitary matrix

$$\boldsymbol{H}_p(z) = \begin{bmatrix} C_{\text{even}}(z) & C_{\text{odd}}(z) \\ D_{\text{even}}(z) & D_{\text{odd}}(z) \end{bmatrix} \quad \text{must have} \quad \begin{array}{rcl} |D_{\text{even}}(z)| & = & |C_{\text{odd}}(z)| \\ |D_{\text{odd}}(z)| & = & |C_{\text{even}}(z)|. \end{array}$$

Go further to show that $D_{\text{even}} = \pm z^{-2L}$ (flip of C_{odd}) and $D_{\text{odd}} = \pm z^{-2L}$ (flip of $-C_{\text{even}}$). This gives the alternating flip with any even delay z^{-2L}.

10. Find $H_0(z)$ and $H_1(z)$ and $\boldsymbol{H}_p(z)$. Is this paraunitary? Find the PR synthesis filters. What is $\boldsymbol{F}_p(z)$?



11. For $\boldsymbol{H}_p(z) = \boldsymbol{I} - \boldsymbol{v}\boldsymbol{v}^T + z^{-1}\boldsymbol{v}\boldsymbol{v}^T$ with $\boldsymbol{v} = \frac{1}{3}\,[\ 2 - 1 \quad 1 \quad \sqrt{3}\ ]^T$, find the four PR filters.

12. Let $\boldsymbol{H}_p(z) = \boldsymbol{R}_1\Lambda(z)\boldsymbol{R}_0$ where $\theta_1 = \frac{\pi}{4}$ and $\theta_0 = -\frac{\pi}{2}$. What are the analysis and synthesis filters? Plot the frequency responses of $H_k(z)$.

5.3 Halfband Filters

Out of all the equations in the previous section, we would like to emphasize one. It came at the end. It applied first of all to the lowpass filter $C(z)$, and then by the alternating flip also to $D(z)$. It was equation (5.28), that the frequency response $C(\omega) = \sum c(k)e^{-jk\omega}$ satisfies

$$|C(\omega)|^2 + |C(\omega+\pi)|^2 = 2. \tag{5.31}$$

The key question is, *what does equation* (5.31) *say about* $|C(\omega)|^2$ *itself?*

We assign the symbol $P(\omega)$ to this important quantity $|C(\omega)|^2$. It is the ***power spectral response***. Because $C(\omega)$ multiplies $\overline{C(\omega)}$, the filter with this response $P(\omega)$ is *symmetric*. It is the "autocorrelation filter":

$$P(\omega) = \sum_{-N}^{N} p(n)e^{-jn\omega} = \Big(\sum_{0}^{N} c(k)e^{-jk\omega}\Big)\Big(\sum_{0}^{N} c(k)e^{jk\omega}\Big). \tag{5.32}$$

The function $P(\omega) = |C(\omega)|^2$ is real and nonnegative. It equals its complex conjugate. This verifies the symmetry $p(n) = p(-n)$ that was expected (with real coefficients).

To repeat: When $\sum c(k)e^{-jk\omega}$ multiplies its conjugate $\sum c(l)e^{jl\omega}$, we watch for $n = k - l$ which is $l = k - n$. The coefficient $p(n)$ is the sum of $c(k)$ times $c(k-n)$:

$$p(n) = \sum c(k)c(k-n) = \text{autocorrelation of the sequence } c(k). \tag{5.33}$$

Autocorrelation is $\boldsymbol{p} = \boldsymbol{c} * \boldsymbol{c}^T$. This is the convolution of $\boldsymbol{c} = (c(0), c(1), c(2), \ldots)$ with its time reversal $\boldsymbol{c}^T = (\ldots, c(2), c(1), c(0))$. Replacing $-n$ by n in (5.33) brings no change in $\boldsymbol{p}$.

The reason for the surprising notation c^T is that multiplying $C(\omega)$ by $\overline{C(\omega)}$ corresponds exactly to multiplying the infinite filter matrix C by C^T:

$$P = CC^T = \begin{bmatrix} \cdot & & & \\ & \cdot & & \\ \cdot & \cdot & & \\ & \cdot & c(0) & \\ \cdot & c(1) & & \cdot \\ c(2) & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c(2) & \\ \cdot & c(1) & & \cdot \\ c(0) & & \cdot & \\ & \cdot & & \cdot \\ & & \cdot & \\ & & & \cdot \end{bmatrix}. \tag{5.34}$$

P is a symmetric positive definite (or semidefinite) Toeplitz matrix. The transpose of CC^T is CC^T. The diagonal $p(0) = c(0)^2 + c(1)^2 + c(2)^2 + \cdots$ is certainly positive. Equation (5.31) says that $p(0) = 1$, when the matrix C comes from an orthonormal filter bank.

What does equation (5.31) say about the other coefficients $p(n)$? In a word, it says *nothing* about the odd coefficients and it assigns *zero* to the even coefficients. C is the start of an orthogonal filter bank if and only if the autocorrelation filter P is a ***halfband filter***:

$$P(z) + P(-z) = 2. \tag{5.35}$$

The *even coefficients* with $n = 2m$ must be $\delta(m)$:

$$\textbf{Halfband filter} \quad p(2m) = \sum c(k)c(k-2m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0. \end{cases} \tag{5.36}$$

The odd coefficients are not necessarily zero! They cancel automatically in equation (5.35). To require $CC^T = I$, with odd coefficients zero, would make C an allpass filter. It could only be a delay. The requirement $P(z) + P(-z) \equiv 2$ is much weaker than $P(z) \equiv 1$. Perfect reconstruction is possible for an FIR filter bank, but essentially impossible for one FIR filter.

The highpass response $D(\omega)$ leads similarly to the autocorrelation $P_1(\omega) = |D(\omega)|^2$. Then DD^T must also be a normalized halfband filter. That result is automatic with the alternating flip, which gives $P_1(\omega) = P(\omega + \pi)$. Then $p_1(2m) = p(2m) = \delta(m)$ for the even coefficients. The odd coefficients change sign, $P_1(z) = P(-z)$, but the halfband condition $P_1(z) + P_1(-z) = 2$ remains true.

The sum $|C(\omega)|^2 + |C(\omega+\pi)|^2$ is $P(\omega) + P(\omega+\pi)$. For a halfband filter, this sum is a constant. The graph of $P(\omega)$ shows a special symmetry with respect to the halfband frequency $\omega = \frac{\pi}{2}$ — hence the name. Notice what happens in downsampling — the even coefficients yield the identity filter:

$$(\downarrow 2)P = I \quad \text{when } P \text{ is normalized halfband.} \tag{5.37}$$

Example 5.1. The symmetric filter with $p(0) = 1$ and $p(1) = p(-1) = \frac{1}{2}$ is a halfband filter. Its response is

$$P(z) = 1 + \frac{z^{-1} + z}{2} \text{ and equivalently } P(\omega) = 1 + \cos\omega.$$

In the z-domain, $P(z) + P(-z) = 2$. The odd powers cancel and there is no z^2 term. In the ω-domain $1 + \cos\omega + 1 - \cos\omega = 2$. The odd frequency cancels. *Notice that* $P(\omega) = 1 + \cos\omega$

is never negative! It does reach zero at the highest frequency $\omega = \pi$, corresponding to $z = -1$. When $P(\omega)$ touches zero, its spectral factor $C(\omega)$ must also touch zero — since $P = |C|^2$. This leads us to $C(\omega) = (1 + e^{-i\omega})/\sqrt{2}$:

$$P(\omega) = |C(\omega)|^2 = \left(1 + e^{-i\omega}\right)\left(1 + e^{i\omega}\right)/2 = 1 + \cos\omega. \tag{5.38}$$

These coefficients $c(0) = c(1) = 1/\sqrt{2}$ come from the familiar averaging filter. The division by $\sqrt{2}$ gives an ortho*normal* filter. Figure 5.2 shows the lowpass halfband filter P with response $1 + \cos\omega$. Added to $P(\omega + \pi)$ it gives the constant 2.

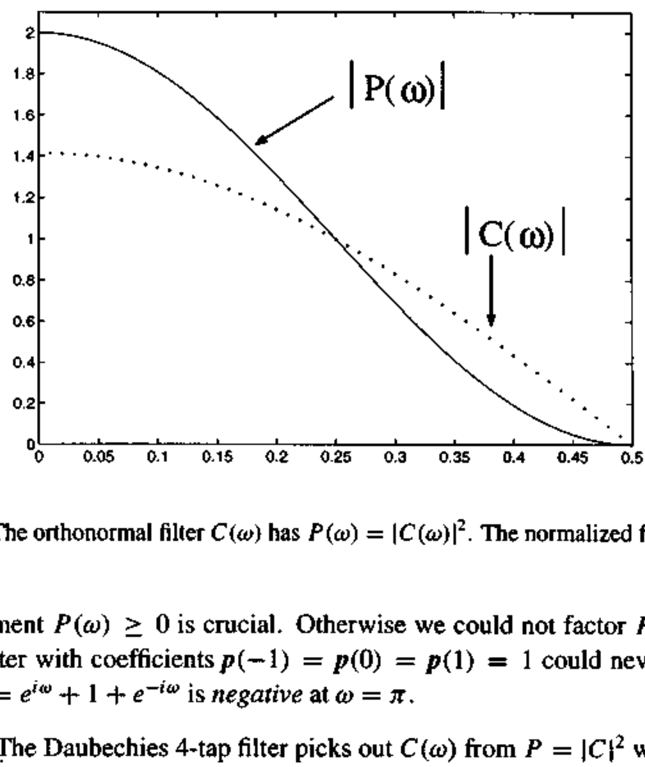


Figure 5.2: The orthonormal filter $C(\omega)$ has $P(\omega) = |C(\omega)|^2$. The normalized frequency 0.5 is $\omega = \pi$.

The requirement $P(\omega) \geq 0$ is crucial. Otherwise we could not factor $P(\omega)$ into $|C(\omega)|^2$. The halfband filter with coefficients $p(-1) = p(0) = p(1) = 1$ could never be $|C(\omega)|^2$. Its response $P(\omega) = e^{i\omega} + 1 + e^{-i\omega}$ is *negative* at $\omega = \pi$.

Example 5.2. The Daubechies 4-tap filter picks out $C(\omega)$ from $P = |C|^2$ when

$$P(\omega) = (1 + \cos\omega)^2\left(1 - \tfrac{1}{2}\cos\omega\right). \tag{5.39}$$

Note the double zero at $\omega = \pi$, coming from $(1 + \cos\omega)^2$. If we keep only that factor, this $P(\omega)$ would be the square of the previous example. Its factor would be the square of the previous $C(\omega)$, namely $\frac{1}{2} + e^{-i\omega} + \frac{1}{2}e^{-2i\omega}$. Those coefficients $\frac{1}{2}, 1, \frac{1}{2}$ are important — they will lead to the *hat function*, when we study wavelets. But $P(\omega) = (1 + \cos\omega)^2$ does not by itself yield a halfband filter, so the lowpass C with coefficients $\frac{1}{2}, 1, \frac{1}{2}$ cannot go into an orthogonal filter bank. The hat function is not orthogonal to its translates.

To repeat: $(1 + \cos\omega)^2$ includes the term $\cos^2\omega$. This produces an even frequency $\cos 2\omega$. In the z-domain we are squaring $1 + \frac{1}{2}\left(z^{-1} + z\right)$, which produces the even power z^{-2}. This is not halfband! Daubechies' extra factor $1 - \frac{1}{2}\cos\omega$ must be included, to cancel the 2ω term in $P(\omega)$ and the z^2 term in $P(z)$. We will have orthogonality, thanks to that factor.

To see that $P(\omega)$ is halfband, multiply it out. The $\cos 2\omega$ term is missing:

$$P(\omega) = \left(1 + 2\cos\omega + \cos^2\omega\right)\left(1 - \tfrac{1}{2}\cos\omega\right) = 1 + \tfrac{3}{2}\cos\omega - \tfrac{1}{2}\cos^3\omega.$$

In the z-domain, the z^2 term is missing. Its coefficient $p(2)$ is zero:

$$P(z) = -\tfrac{1}{16}z^3 + \tfrac{9}{16}z + 1 + \tfrac{9}{16}z^{-1} - \tfrac{1}{16}z^{-3}. \tag{5.40}$$

We know that $P(\omega) \geq 0$ because $1 - \frac{1}{2}\cos\omega > 0$. Therefore it can be factored into $|C(\omega)|^2$ (*spectral factorization*). This is not a trivial calculation. It is made easier by the fact that we already know the factor for $1 + \cos\omega$, from the first example. This leaves only the linear piece $Q(\omega) = 1 - \frac{1}{2}\cos\omega$ or $Q(z) = 1 - \frac{1}{4}\left(z^{-1} + z\right)$. We factor Q in three steps:

$$1 - \tfrac{1}{4}\left(z^{-1} + z\right) = \left(b(0) + b(1)z^{-1}\right)\left(b(0) + b(1)z\right) \tag{5.41}$$

$$1 = b(0)^2 + b(1)^2 \quad \text{and} \quad -\tfrac{1}{4} = b(0)b(1) \tag{5.42}$$

Solving the quadratic equations gives $b(0) = (1 + \sqrt{3})/\sqrt{8}$ and $b(1) = (1 - \sqrt{3})/\sqrt{8}$. The solutions are real because $1 - \frac{1}{2}\cos\omega$ is safely positive.

Another approach, basically the same, is to multiply (5.41) by z to get an ordinary quadratic. The quadratic formula gives its roots as $2 \pm \sqrt{3}$. Since $Q = 0$ at these roots, we have

$$b(0) + b(1)\,(2 \pm \sqrt{3})^{-1} = 0. \tag{5.43}$$

The previous solution is correct because $1 + \sqrt{3} + (1 - \sqrt{3})(2 - \sqrt{3})^{-1} = 0$. This is the *minimum phase solution*, from the root $2 - \sqrt{3}$ inside the unit circle. There is another solution from $2 + \sqrt{3}$, in which $b(0)$ and $b(1)$ are exchanged. This is *maximum phase*. Two more solutions come from reversing signs to $-b(0)$ and $-b(1)$. We are seeing the limited number of possible spectral factors in $|C(\omega)|^2$. The general rule for higher degree polynomials and longer filters is the same:

Minimum phase: Choose roots of $z^N P(z)$ that are on or *inside* $|z| = 1$.

Maximum phase: Choose roots of $z^N P(z)$ that are on or *outside* $|z| = 1$.

Mixed phase is also possible, choosing some roots inside and some outside. That can bring us to linear phase. We will show how linear phase factors of the Daubechies polynomials lead to *biorthogonal filter banks* which are among the current favorites.

Completion of Example. We factored $1 - \frac{1}{2}\cos\omega$ and $1 + \cos\omega$. Multiply to obtain

$$\begin{aligned} C(z) &= \tfrac{1}{4\sqrt{2}}\left(1 + z^{-1}\right)^2\left(\left(1 + \sqrt{3}\right) + \left(1 - \sqrt{3}\right)z^{-1}\right) \\ &= \tfrac{1}{4\sqrt{2}}\left[\left(1 + \sqrt{3}\right) + \left(3 + \sqrt{3}\right)z^{-1} + \left(3 - \sqrt{3}\right)z^{-2} + \left(1 - \sqrt{3}\right)z^{-3}\right]. \end{aligned} \tag{5.44}$$

Those are the four coefficients $c(0), \ldots, c(3)$ of the famous Daubechies filter D_4. In our present normalization, they are divided by $\sqrt{32} = 4\sqrt{2}$. In other normalizations they are divided by 4. Remember their two key properties:

1. The halfband filter has $P(z) + P(-z) = 2$. The factor $C(z)$ goes into an orthonormal filter bank. $D(z)$ comes from $C(z)$ by an alternating flip.

2. The response $C(z)$ has a double zero at $z = -1$. In frequency, $C(\omega)$ has a double zero at $\omega = \pi$. The response is *flat* at π because of $(1 + \cos\omega)^2$.

The double zero at $\omega = \pi$ will produce *two vanishing moments* for the Daubechies wavelets in Chapter 6.

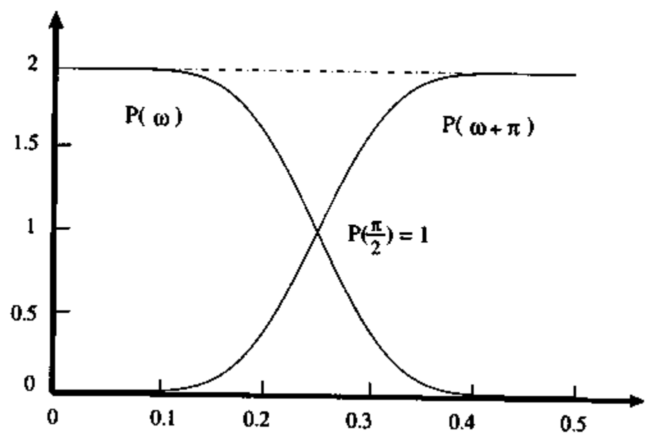


Figure 5.3: A halfband filter $P(\omega)$ and its mirror image $P(\omega + \pi)$. Their sum is constant.

Figure 5.3 shows a graph of $|C(\omega)|^2$, so you can see the halfband property that produces orthogonality. The highpass response in that figure is $|C(\omega + \pi)|^2$, which equals $|D(\omega)|^2$. The sum of the two is constant, so there is no amplitude distortion. The flatness gives great accuracy near $\omega = 0$ and $\omega = \pi$ (not so great in the middle). The filter bank gives perfect reconstruction.

Problem Set 5.3

1. Solve equations (5.42) for $x = b(0)^2$. Confirm the factorization.
2. What is the 2 by 2 polyphase matrix $\boldsymbol{H}_p(z)$ from the Daubechies $C(z)$ and the alternating flip?
3. What is the 2 by 2 modulation matrix $\boldsymbol{H}_m(z)$ for that four-tap Daubechies example? Verify that $\widetilde{\boldsymbol{H}}_m \boldsymbol{H}_m = 2\boldsymbol{I}$.
4. If a linear phase halfband filter satisfies $G(z) + G(-z) = z^{-l}$, what is the relation between l and N? Can $G(z)$ be an antisymmetric?

5.4 Spectral Factorization

In an orthonormal filter bank, $C(z) = \sqrt{2}H(z)$ is a *spectral factor* of a symmetric halfband filter $P(z)$. The factorization is $P(z) = C(z^{-1})C(z)$ and the halfband property is $P(z) + P(-z) = 2$. In frequency, $P(\omega) = |C(\omega)|^2$ achieves the orthogonality condition $|C(\omega)|^2 + |C(\omega+\pi)|^2 = 2$. In the reverse direction, $P(z)$ is the *autocorrelation* of $C(z)$. This intimate relation of spectral factor $C(z)$ and its autocorrelation $P(z)$ is fundamental throughout signal processing.

Two questions arise immediately:

1. (Theory) Can every polynomial with $P(\omega) \geq 0$ be factored into $|C(\omega)|^2$?

2. (Practice) How is this spectral factorization actually done?

The answer to Question 1 is *yes*. This is the Féjer-Riesz Theorem. The answer to Question 2 is not so quick. There are many competing algorithms for spectral factorization. Short filters offer no serious difficulty, but with 100 or even 50 coefficients the weaker algorithms become slow and/or unreliable. When $C(\omega)$ is only approximate, the reconstruction is not perfect.

The trigonometric polynomials $P(\omega)$ and $C(\omega)$ are both of degree N:

$$\sum_{-N}^{N} p(n)e^{-in\omega} = |C(e^{i\omega})|^2 = \left|\sum_{0}^{N} c(n)e^{-in\omega}\right|^2.$$

$P(z)$ has symmetric coefficients $p(n) = p(-n)$. There are $N+1$ independent coefficients in P and the same number in C. They are linked by quadratic equations, when we solve $P(\omega) = |C(\omega)|^2$. Those equations are solvable if and only if $P(\omega) \geq 0$ for all ω.

As an aside, note that *matrix* spectral factorization is also possible where $P(\omega)$ is symmetric positive definite. Both 1 and 2, theory and practice, are nontrivial. The Riccati equation is involved.

We indicate four factorization methods. Three are actually used; Method C is for conversation only. The first method begins by finding the zeros of a polynomial (by a good algorithm!). This proves that spectral factorization is possible, by doing it.

Method A (zeros of a polynomial). With real symmetric coefficients $p(n)$, we have $P(z) = P(1/z)$. ***If*** z_i ***is a root, so is*** $1/z_i$. When z_i is inside the unit circle, $1/z_i$ is outside. The roots z_j *on* the unit circle must have even multiplicity, by the crucial assumption that $P(\omega) \geq 0$. Therefore the polynomial $z^N P(z)$ of degree $2N$, with leading coefficient $p(N) \neq 0$, must have these $2N$ factors:

$$z^N P(z) = p(N) \prod_{i=1}^{M} (z - z_i)\left(z - \frac{1}{z_i}\right) \prod_{j=1}^{N-M} \left(z - z_j\right)^2. \tag{5.45}$$

This contains the key point, but we know more. Real coefficients ensure that the complex conjugate $\overline{z}$ is a root when z is a root. The complex roots off the unit circle actually come *four at a time*: z_i and $\overline{z}_i$ inside, $1/z_i$ and $1/\overline{z}_i$ outside. The complex roots on the circle also come four at a time: z_j twice and $\overline{z}_j$ twice. Real roots on the circle come two at a time (even multiplicity).

Now construct $C(z)$ by taking *all* the roots z_i (including $\overline{z}_i$) inside the circle, and also take one out of every double root z_j on the circle:

$$z^N C(z) = |p(N)|^{1/2} \prod_{i=1}^{M} (z - z_i) \prod_{j=1}^{N-M} \left(z - z_j\right). \tag{5.46}$$

This is the "minimum phase spectral factor." It has no roots outside the circle. The coefficients of $C(z)$ are still real, because the complex roots are automatically in conjugate pairs: $\overline{z}_i$ and $\overline{z}_j$ came with z_i and z_j.

Example 5.3. The 4-tap Daubechies filter in the previous section led to zeros at $z_i = 2 - \sqrt{3}$ and $z_i^{-1} = 2 + \sqrt{3}$. The other four roots of $z^3 P(z)$ are at $z_j = -1$ (on the unit circle and again real). Two of those roots go into the spectral factor (5.46). Thus $z^3 C(z)$ is a cubic polynomial with roots $2 - \sqrt{3}$, -1, and -1. It is minimum phase.

Every factorization of $P(z)$ into $F(z)\,H(z)$ must put some roots into $H(z)$ and the remaining roots into $F(z)$. The rules for this separation of roots of $P(z)$ are:

- For F and H to be ***real*** filters, z and $\overline{z}$ must stay together.
- For F and H to be ***symmetric*** filters, z and z^{-1} must stay together.

- For F to be the transpose of H, z and z^{-1} must go separately. This is the spectral factorization $C(z)\,C(z^{-1})$ that gives an orthogonal filter bank when P is halfband.

Figure 5.4a shows a partition of the zeros into circles and squares that makes both factors symmetric. The splitting in Figure 5.4b makes one factor the transpose (coefficients reversed) of the other factor. *To achieve both properties at the same time, all zeros of $P(z)$ – not just the zeros on the unit circle – must be of even multiplicity.* We now show that this is impossible for a halfband filter. This gives another proof of Theorem 5.3, that orthogonality conflicts with symmetry.

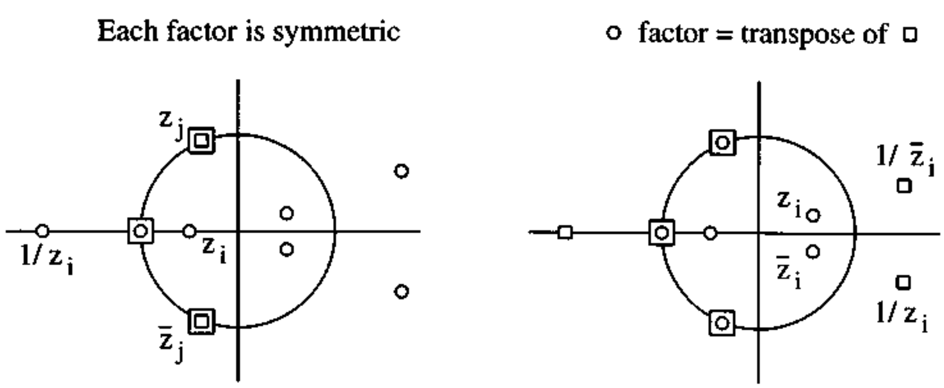


Figure 5.4: Twelve zeros of a halfband filter separate into analysis (○) and synthesis (□).

Theorem 5.4 *A symmetric orthogonal FIR $C(z)$ can only have two nonzero coefficients.*

Proof. An FIR filter is symmetric when $z^N C(z) = C(z^{-1})$. If z_i is a zero of $C(z)$, so is z_i^{-1}. Then $P(z) = C(z^{-1})C(z)$ has a *double root* at z_i. More precisely, all roots of the polynomial $z^N P(z)$ have even multiplicity. This polynomial is a perfect square $[R(z)]^2$.

If the filter is also orthogonal, $P(z)$ must be halfband:

$$z^N P(z) = [r(0) + \cdots + r(N)z^N]^2 \quad \text{has only one odd power } z^N.$$

The first odd power in $R(z)$ produces (when it multiplies $r(0)$) an odd power in $[R(z)]^2$. The last even power in $R(z)$ also produces (when it multiplies $r(N)\,z^N$) an odd power in $[R(z)]^2$. But our halfband filter has only one odd power. *We cannot allow any even powers or any odd powers in the terms $\cdots$ indicated by the three dots.* The polynomial $R(z)$ has only two terms and $P(z)$ has three terms. Then $C(z)$ only has two nonzero coefficients.

All symmetric orthogonal FIR filters have $C(z) = (1+z^{-N})/\sqrt{2}$ with odd N. The halfband product filter is $P(z) = \frac{1}{2}z^N + 1 + \frac{1}{2}z^{-N}$.

Example 5.4. For symmetric filters, the roots $z_i = 2-\sqrt{3}$ and $z_i^{-1} = 2+\sqrt{3}$ must stay together when we factor $P(z)$. The four roots at $z = -1$ can be split between $H(z)$ and $F(z)$. One symmetric splitting is $H(z) = (1 + z^{-1})(-1 + 4z^{-1} - z^{-2})/2\sqrt{2}$ and $F(z) = (1+z^{-1})^3/4\sqrt{2}$. There are several symmetric factorizations, but none of them can be orthogonal.

Now we return to **computation of zeros of polynomials**. For long filters, a good algorithm is needed to find the zeros z_i and z_j of $P(z)$. We quote from the 1994 abstract by Lang and Frenzel [La,Fr]:

> Finding polynomial roots rapidly and accurately is an important problem in many areas of signal processing. We use Müller's method for computing a root of the deflated polynomial. This estimate is improved by applying Newton's method to the original polynomial. Furthermore we give a simple approach to improve the accuracy for spectral factorization when there are double roots on the unit circle.

Müller's method uses three previous estimates of the root of $z^N P(z)$ to find the next estimate. The parabola that interpolates at the three old points has a root at the new point. Since parabolas can have complex roots, Müller's algorithm can find complex roots from a real start — while Newton can only move chaotically on the real line.

Newton's method uses the most recent estimate z_k. For real roots, the tangent line at z_k to the graph of $z^N P(z)$ crosses zero at the new point z_{k+1}. This is the outstanding method for solving nonlinear equations, provided z_0 is close enough — which is the task of Müller's method. The polynomial $z^{10000} - 1$ was one of the tests (not the only one!). The code is on ftp from *cml.rice.edu* under directory *pub/software*.

MATLAB ***uses an eigenvalue method.*** Its subroutine ***roots*** is effective up to quite large degree. The roots of a polynomial $z^N + \cdots$ are the eigenvalues of its $N \times N$ *companion matrix*, which has 1's down a diagonal and minus the polynomial coefficients along a row. For example, $z^3 - 2z^2 - 5z - 9$ is specified by the vector $\boldsymbol{v} = [1 \;\; -2 \;\; -5 \;\; -9]$. The command $\boldsymbol{M} = \text{compan}(\boldsymbol{v})$ produces the matrix $\boldsymbol{M} = \begin{bmatrix} 2 & 5 & 9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ with $\det(z\boldsymbol{I} - \boldsymbol{M}) = z^3 - 2z^2 - 5z - 9$. MATLAB finds the eigenvalues (by the QR method) which are the roots of the polynomial.

The next section mentions how rescaling y to $4y$ allowed us to compute Daubechies filters of twice the length achievable without this scaling. The coefficients in $P(z)$ were better controlled. Linear phase filters with extremely good stopband attenuation have many zeros on or near the unit circle. These are the hardest zeros to compute.

Method B (solve quadratic equations). We are looking for $N + 1$ numbers $c(0), \ldots, c(N)$. The $N + 1$ equations are of second degree, involving c's times c's. The equations come from matching powers of $e^{i\omega}$ in $\overline{C(\omega)}C(\omega) = P(\omega)$:

$$\left(\sum_0^N c(k)e^{ik\omega}\right)\left(\sum_0^N c(k)e^{-ik\omega}\right) = \sum_{-N}^N p(n)e^{-in\omega}. \tag{5.47}$$

One way to make those equations explicit is in matrix form:

$$\begin{bmatrix} c(0) & c(1) & c(2) & \cdot & c(N) \\ & c(0) & c(1) & c(2) & \cdot \\ & & c(0) & c(1) & c(2) \\ & & & c(0) & c(1) \\ & & & & c(0) \end{bmatrix} \begin{bmatrix} c(0) \\ c(1) \\ c(2) \\ \cdot \\ c(N) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ \cdot \\ p(N) \end{bmatrix}. \tag{5.48}$$

The first equation is $c(0)^2 + \cdots + c(N)^2 = p(0)$. This gives the constant term in (5.47). The second equation gives the $e^{-i\omega}$ term. The last equation is $c(0)c(N)e^{-iN\omega} = p(N)\,e^{-iN\omega}$.

Equation (5.48) is not a linear system! It is quadratic, in fact homogeneous of degree 2. It has a real solution if and only if $P(\omega) \geq 0$ for all ω. This is not an easy condition to verify on the coefficients $p(n)$. If $P(\omega) \geq 0$ is not true — in which case our solution methods must fail — we can add enough to the DC term $p(0)$ to make it true.

For orthogonality, the $p(n)$ come from a halfband filter. The even coefficients $p(2)$, $p(4)$, ... are all zero. But our discussion is not in any way limited to this halfband case. Spectral factorization applies to all filters with $P(\omega) \geq 0$. It even applies to IIR filters, but those lead to infinitely many equations.

Example 5.5. The previous section factored $P(\omega) = 1 - \frac{1}{4}\left(e^{-i\omega} + e^{i\omega}\right)$. Comparing coefficients of 1 and $e^{-i\omega}$ led us to

$$\begin{aligned} c(0)^2 + c(1)^2 &= 1 \\ c(0)c(1) &= -\tfrac{1}{4} \end{aligned} \quad \text{or} \quad \begin{bmatrix} c(0) & c(1) \\ 0 & c(0) \end{bmatrix} \begin{bmatrix} c(0) \\ c(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix}. \qquad (5.49)$$

This is our system (5.48) for that particular example with $N = 1$. Eliminating $c(1)$ gave a single quadratic equation. The unknown was $x = c(0)^2$ and the equation was $x + \frac{1}{16x} = 1$ or $16x^2 - 16x + 1 = 0$. For $N > 1$ we *cannot* reduce the $N + 1$ quadratic equations to a single equation for $c(0)$. An approximate solution by method A, B, C, or D (or another method E) is the best we can expect.

To use a nonlinear equation solver, write the kth quadratic equation as $\frac{1}{2}c^T Q(k) c = p(k)$. The symmetric matrix $Q(k)$ has 1's along its kth subdiagonal and superdiagonal (and $Q(0) = 2I$). Here is $Q(2)$ with $N = 3$:

$$\frac{1}{2}\begin{bmatrix} c(0) & c(1) & c(2) & c(3) \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c(0) \\ c(1) \\ c(2) \\ c(3) \end{bmatrix} = p(2). \qquad (5.50)$$

This is $c(0)c(2) + c(1)c(3) = p(2)$ from matching the $e^{-2i\omega}$ terms in $|C(\omega)|^2 = P(\omega)$. The partial derivatives of that left side $L(2)$ are in the gradient vector $Q(2)c$:

$$\begin{aligned} \partial L(2)/\partial c(0) &= c(2) \\ \partial L(2)/\partial c(1) &= c(3) \\ \partial L(2)/\partial c(2) &= c(0) \\ \partial L(2)/\partial c(3) &= c(1) \end{aligned} \quad \text{agrees with} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c(0) \\ c(1) \\ c(2) \\ c(3) \end{bmatrix}.$$

The kth quadratic function is $L(k) = \frac{1}{2}c^T Q(k) c$, and its gradient is $Q(k)c$.

The second derivatives are also desired by nonlinear subroutines, and also readily available. They are in the constant matrix $Q(k)$. One successful program using gradients and second derivatives has been the Quadratic Constrained Least Squares (QCLS) optimization code by anonymous ftp from *eceserv0.ece.wisc.edu* under the directory *pub/nguyen/software/QCLS*. We use it in Chapter 9 to design M-band filter banks.

Method C (matrix factorization). Equation (5.48) is an attractive form for the quadratic equations. But there is a more symmetric form. If you like infinite matrices, you will enjoy this. Instead of a finite matrix times a vector, it has an infinite constant-diagonal matrix C times its transpose:

$$C^TC = \begin{bmatrix} c(0) & c(1) & \cdot & c(N) & & & \\ & c(0) & c(1) & \cdot & c(N) & & \\ & & c(0) & c(1) & \cdot & c(N) & \\ & & & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} c(0) & & & \\ c(1) & c(0) & & \\ \cdot & c(1) & c(0) & \\ c(N) & \cdot & c(1) & \cdot \\ & c(N) & \cdot & \cdot \\ & & c(N) & \cdot \\ & & & \cdot \end{bmatrix} = P.$$

The columns of P have entries $p(-N), \ldots, p(0), \ldots, p(N)$. In this matrix form of $|C(\omega)|^2 = P(\omega)$, the symmetric matrix P factors into *upper triangular* C^T *times lower triangular* C. This is only possible if P is *positive semi-definite* — which is exactly our condition $P(\omega) \geq 0$.

Matrices are decomposed into triangular factors every day. This is the matrix statement of ordinary Gaussian elimination. The factorization is usually written $A = LU$. It gives *lower triangular times upper triangular*. Fortunately, infinite constant-diagonal matrices commute; our equation is also $CC^T = P$. The harder problem is to factor infinite matrices in finite time.

Approximate method: take a finite section of P. Keep R rows and columns, where R is larger (perhaps much larger) than N. This finite piece P_R is still symmetric and positive definite. Therefore P_R can be factored into $C_RC_R^T$, where C_R is lower triangular. That is the *Cholesky factorization* of P_R. It is a symmetrized form of $A = LU$, available because P_R is symmetric positive definite.

The finite matrix C_R does not contain the exact $c(k)$. It is not even true that C_R has constant diagonals (although P_R has). The reduction to a finite matrix has chopped the tail ends of the row-column multiplications, either in C^TC or in CC^T. But the rows of the computed factor C_R *do approach the rows of* C. The correct (minimum-phase) coefficients $c(k)$ appear in the limit as $R \to \infty$.

We demonstrate with $P(\omega) = 1 - \frac{1}{2}\cos\omega$. This Daubechies example was solved exactly for $c(0) = (1+\sqrt{3})/\sqrt{8} = 0.9659$ and $c(1) = (1-\sqrt{3})/\sqrt{8} = -0.2588$. Take $R = 4$ and use ***chol*** in MATLAB to factor P_4 into CC^T:

$$P_4 = \begin{bmatrix} 1 & -0.25 & & \\ -0.25 & 1 & -0.25 & \\ & -0.25 & 1 & -0.25 \\ & & -0.25 & 1 \end{bmatrix} \quad \text{and}$$

$$C = \begin{bmatrix} 1 & & & \\ -0.25 & 0.9682 & & \\ & -0.2582 & 0.9661 & \\ & & -0.2588 & 0.9659 \end{bmatrix}.$$

This matrix has $c(0)$ and $c(1)$ correct to four places in the last row. But with long filters this method is very slow.

Method D (Cepstral method: Take logarithms). The idea is to convert the multiplication $P(z) = C(z^{-1})C(z)$ into addition. Formally, $\log\left(\sum p(n)z^{-n}\right)$ is easily separated into positive and negative powers of z. The symmetry $p(n) = p(-n)$ and the positivity $P(\omega) \geq 0$ yield a logarithm $L(z)$ with coefficients $l(n) = l(-n)$:

$$\log P(z) = \sum_{-\infty}^{\infty} l(n)z^{-n} = \left(\frac{l(0)}{2} + \sum_{1}^{\infty} l(n)z^{n}\right) + \left(\frac{l(0)}{2} + \sum_{1}^{\infty} l(n)z^{-n}\right).$$

These are infinite series. The logarithm of a polynomial is not a polynomial. This means that our finite computations can only be approximate. The easy separation into $\log C(z^{-1}) + \log C(z)$ is the key advantage of the method.

The sequence $l(n)$ is the *complex cepstrum* of $p(n)$. The series for $L(z)$ converges in an annulus $|z_i| < |z| < 1/|z_i|$ of the complex plane. Here z_i is the largest root of $P(z)$ inside the unit circle, and $1/z_i$ is the smallest root outside. (The method is in trouble with roots z_j *on* the circle. Best to remove those first. Otherwise $L(z) = \log P(z)$ will be infinite at z_j and the series cannot converge.) The computation of the c's requires an inverse Fourier transform of $\log P(z)$. Then the c's are computed from the l's.

A detailed treatment of the cepstrum $l(n)$ is given by Oppenheim and Schafer [OS]. The constant terms give $c(0) = \exp \frac{1}{2}l(0)$. The next term $c(1)$ is interesting because the z^{-1} terms only involve $l(0)$ and $l(1)$. The recursion for $n = 1, 2, \ldots$ turns out to be

$$c(n) = l(n)c(0) + \tfrac{n-1}{n}l(n-1)c(1) + \cdots + \tfrac{1}{n}l(1)c(n-1).$$

We need only $N+1$ coefficients $l(0), \ldots, l(N)$ in the logarithm to find all $N+1$ coefficients $c(0), \ldots, c(N)$ in the spectral factor. To find those $l(n)$ from the given $p(n)$, we use a large-size FFT in the z-domain. A typical size is $8N$, for acceptable accuracy.

This cepstral method does *not* compute zeros of polynomials. So it doesn't find symmetric filters. It is a good way to find orthogonal filters. A code is available by anonymous ftp at *eceserv0.ece.wisc.edu* under directory *pub/nguyen/software/CEPSTRAL*.

Very optional comment. Spectral factorization also solves *singly infinite* constant-diagonal systems. (This is the genuine Toeplitz problem. Doubly infinite matrices could be named after Laurent — but mostly we still say Toeplitz.) The coefficient matrix P_+ has entries $p(i-j)$ only for $i \ge 0$ and $j \ge 0$:

$$P_+x_+ = b_+ \text{ is } \begin{bmatrix} p(0) & p(1) & p(2) & \cdot \\ p(1) & p(0) & p(1) & \cdot \\ p(2) & p(1) & p(0) & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdot \end{bmatrix} = \begin{bmatrix} b(0) \\ b(1) \\ b(2) \\ \cdot \end{bmatrix}. \tag{5.51}$$

This corresponds in continuous time $t \ge 0$ to a *Wiener-Hopf integral equation*:

$$\int_0^\infty p(s-t)x(t)\,dt = b(s) \text{ for } s \ge 0. \tag{5.52}$$

P_+ does not have constant-diagonal factors in $P_+ = LU$. Lower triangular times upper destroys the time-invariant pattern. (Starting at time zero is responsible.) The beautiful Wiener-Hopf idea is that *upper times lower succeeds perfectly*:

$$P_+ = C_+^T C_+ = \begin{bmatrix} c(0) & c(1) & c(2) & \cdot \\ & c(0) & c(1) & \cdot \\ & & c(0) & \cdot \\ & & & \cdot \end{bmatrix} \begin{bmatrix} c(0) & & & \\ c(1) & c(0) & & \\ c(2) & c(1) & c(0) & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{5.53}$$

Wiener and Hopf computed this spectral factorization of P_+ by Method D. One of Norbert Wiener's great theorems is that $\sum |l(n)| < \infty$ when $\sum |p(n)| < \infty$. With no zeros of $P(z)$ on the unit circle, he could take the logarithm even for IIR filters. The solution is

$$x_+ = (P_+)^{-1}b_+ = (C_+)^{-1}(C_+^T)^{-1}b_+. \tag{5.54}$$

The inverses of C_+ and C_+^T are constant-diagonal. So Wiener-Hopf can compute the spectral factorization $P(\omega) = |C(\omega)|^2$ and transform back to the time domain.

Finite constant-diagonal matrices don't have constant-diagonal factors and spectral factorization no longer succeeds. Nevertheless $\boldsymbol{P}_N\boldsymbol{x}_N = \boldsymbol{b}_N$ can be solved quickly by the Levinson algorithm or a "superfast" algorithm or by preconditioned conjugate gradients [ChSt].

We added these comments because transform methods are so central to signal processing. This is the whole underpinning of filter theory.

Problem Set 5.4

1. Suppose $P(z)$ has six zeros at $z = -1$ and four other real zeros at $z = a,\ a^{-1},\ b,\ b^{-1}$. Draw the complex plane and indicate which zeros go into the minimum phase spectral factor $C(z)$.
2. For the same ten zeros, indicate a set of zeros that produces a symmetric (linear phase) $C(z)$. Also indicate a second possibility.
3. Why must all roots of $P(z)$ on the unit circle have even multiplicity, to allow $P(z) = C(z) \times C(z^{-1})$ and $P(\omega) = |C(\omega)|^2$?
4. The coefficients of the Daubechies polynomials $C(z)$ up to order 12 are tabulated at the end of this chapter. Find the zeros using ***roots*** in MATLAB or another algorithm. What were the roots of the halfband polynomial $P(z)$?
5. Show that $P(z) = z^{-N}C(z)C(z^{-1})$ must be symmetric. If C is lowpass with passband and stopband cutoff frequencies ω_p and ω_s and errors δ_p and δ_s, what are the cutoff frequencies and the errors of P?

5.5 Maxflat (Daubechies) Filters

This section is about an important family of filters, which will lead to an outstanding family of wavelets. The same construction yields both. Wavelets come from filters with special properties. Historically, their close relation was not immediately seen — now it is the subject of Chapter 6. The importance of this special construction is in its combination of two key properties:

1. These particular filters (and wavelets) are ***orthogonal***.
2. The frequency responses have ***maximum flatness*** at $\omega = 0$ and $\omega = \pi$.

The lowpass filters will have $p = 1, 2, 3, 4, \ldots$ zeros at π. They have $2p = 2, 4, 6, 8, \ldots$ coefficients, so that $N = 2p - 1$. We use ***boldface p*** for the coefficients of $P(\omega) = |C(\omega)|^2$ and *lightface* p to count the zeros of $C(\omega)$ at $\omega = \pi$. The highpass coefficients $\boldsymbol{d}(k)$ come from an alternating flip. The first member of this family was the subject of Chapter 1: $c(0) = c(1) = 1/\sqrt{2}$. Note the normalization $c(0)^2 + c(1)^2 = 1$. These numbers go into a *unitary matrix*. For each $p = 1, 2, 3, 4, \ldots$ the filter bank is orthonormal. The product filters have degree $2N = 4p - 2$:

$$P_0(z) = \left(\frac{1+z^{-1}}{2}\right)^{2p} Q_{2p-2}(z) \quad \textit{will be halfband by special choice of} \quad Q.$$

In the literature on filters, this family is described as *maxflat*. The coefficients were given by [Herrmann]. They were already in formulas for interpolation, described below. In the history of wavelets, we are reproducing the great 1988 discovery by Ingrid Daubechies. The filters are FIR with $2p$ coefficients. The wavelets are supported on the interval $[0, N] = [0, 2p - 1]$. As p increases, the filters are increasingly "regular" and the wavelets are increasingly "smooth."

This section concentrates on filter properties and coefficients. We give a simple derivation of $P(z)$, and new facts about its zeros. The next chapters will concentrate on wavelets and the step into continuous time.

Condition O and Condition A_p

Before starting, it is helpful to count the requirements we must impose. There are $2p$ numbers to be chosen. These can be the coefficients $c(0), \ldots, c(2p-1)$ in the lowpass filter, with frequency response $C(\omega)$. They could equally well be the coefficients $p(0), \ldots, p(2p-1)$ of the centered (even) polynomial $P(\omega) = |C(\omega)|^2$. The c's come from the p's by spectral factorization. The nonnegative polynomial $P(\omega)$ is factored by the methods of the previous section:

$$P(\omega) = \sum_{1-2p}^{2p-1} p(n)e^{-in\omega} \text{ equals } |C(\omega)|^2 = \left|\sum_{0}^{2p-1} c(n)e^{-in\omega}\right|^2. \tag{5.55}$$

Our formulas yield the numbers $p(n) = p(-n)$. Except for the first few filters in the family, there are no simple formulas for $c(n)$.

These $2p$ numbers are determined by p conditions for orthogonality from Condition O, and p conditions for a flat response from Condition A. More precisely, the requirement is "Condition A_p" — the subscript indicates the order of flatness at $\omega = \pi$ (and $\omega = 0$). Here are the $p + p$ conditions:

Condition O $P = |C|^2$ is a normalized halfband filter:

$$p(0) = 1 \text{ and } p(2) = p(4) = \cdots = p(2p-2) = 0. \tag{5.56}$$

Condition A_p $C(\omega)$ has a zero of order p at $\omega = \pi$:

$$C(\pi) = C'(\pi) = \cdots = C^{(p-1)}(\pi) = 0. \tag{5.57}$$

The equation $C(\pi) = 0$ says that $\sum c(n)(-1)^n = 0$. The odd-numbered coefficients have the same sum as the even-numbered coefficients:

$$\text{Condition } A_1 \text{ on } c(n): \quad \sum_{\text{odd } n} c(n) = \sum_{\text{even } n} c(n). \tag{5.58}$$

This is the first of the "sum rules." Altogether we can impose the pth order zero in (5.57) as p sum rules on the coefficients:

$$\text{Condition } A_p \text{ on } c(n): \quad \sum_{n=0}^{2p-1} (-1)^n n^k c(n) = 0 \quad \text{for } k = 0, 1, \ldots, p-1. \tag{5.59}$$

The factor n^k comes from the kth derivative of $\sum c(n)e^{-in\omega}$. Then $(-1)^n$ comes from substituting $\omega = \pi$. The convention for n^0 is 1.

Note on $C(0) = \sqrt{2}$: The sum rule (5.58) also applies to the coefficients $p(n)$, because $P(\omega) = |C(\omega)|^2$ also vanishes at $\omega = \pi$. The odd sum must be 1, since the only nonzero even-numbered coefficient is $p(0) = 1$:

$$\sum_{\text{odd } n} p(n) = \sum_{\text{even } n} p(n) = p(0) = 1. \tag{5.60}$$

The sum over all n is $P(0) = 2$. Then $P(\omega) = |C(\omega)|^2$ yields $C(0) = \pm\sqrt{2}$. We always choose the plus sign for a lowpass filter, so the DC term at $\omega = 0$ is not reversed in sign:

$$\sum_{\text{all } n} c(n) = C(0) = \sqrt{P(0)} = \sqrt{2}. \tag{5.61}$$

The p zeros at π mean that $C(\omega)$ *has a factor* $\left(1 + e^{-i\omega}\right)^p$:

$$\text{Condition } A_p \text{ on } C(\omega): \quad C(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^p R(\omega). \tag{5.62}$$

$R(\omega)$ has degree $p - 1$, to bring the total degree of $C(\omega)$ to $2p - 1$. You could say that the pth order flatness is accounted for by $\left(1 + e^{-i\omega}\right)^p$. Then the p coefficients in $R(\omega)$ are chosen to satisfy the p equations of Condition O.

To repeat: p equations for orthogonality and p equations for flatness. Condition O is applied to $P(\omega)$; it must be halfband. Condition A_p is applied to $C(\omega)$; it must have the factor $(1 + e^{-i\omega})^p$. This is easily converted to a condition on $P(\omega) = |C(\omega)|^2$, when we use $\left|1 + e^{-i\omega}\right|^2 /2 = (1 + \cos\omega)$:

$$\text{Condition } A_p \text{ on } P(\omega): \quad P(\omega) \text{ has a factor } \left(\frac{1 + \cos\omega}{2}\right)^p. \tag{5.63}$$

Formulas for $P(\omega)$

We intend to give two formulas for $P(\omega) = |C(\omega)|^2$. The one associated with Ingrid Daubechies has $(1 + \cos\omega)^p$ times a sum of p terms. The formula associated with Yves Meyer gives the derivative of $P(\omega)$ as $-c(\sin\omega)^{2p-1}$. Then integration determines c and $P(\omega)$.

The best starting point is the ordinary polynomial $B_p(y)$. This has degree $p - 1$, with p coefficients. It is the binomial series for $(1 - y)^{-p}$, truncated after p terms:

$$B_p(y) = 1 + py + \frac{p(p+1)}{2} y^2 + \cdots + \binom{2p-2}{p-1} y^{p-1} = (1 - y)^{-p} + O(y^p). \tag{5.64}$$

The coefficient of y^k is $\binom{p+k-1}{k}$. The remainder has order y^p because this is the first term to be dropped. The complex zeros of this polynomial $B_p(y)$ will be all-important for the Daubechies filters.

We combine $B_p(y)$ with the factor $(1 - y)^p$ that has p zeros at $y = 1$. The variable y on $[0, 1]$ will correspond to the frequency ω on $[0, \pi]$. The product $\widetilde{P}(y) = 2(1 - y)^p B_p(y)$ has exactly the flatness we want at $y = 0$:

$$2(1 - y)^p B_p(y) = 2(1 - y)^p[(1 - y)^{-p} + O(y^p)] = 2 + O(y^p). \tag{5.65}$$

This is a polynomial of degree $2p-1$. It is the unique polynomial with $2p$ coefficients that satisfies p conditions at each endpoint:

$\widetilde{P}(y)$ ***and its first*** $p-1$ ***derivatives are zero at*** $y=0$ ***and*** $y=1$, ***except*** $\widetilde{P}(0)=2$.

Two more properties follow quickly. First, the derivative has $p-1$ zeros at both end points. It is a polynomial of degree $2p-2$ and with those zeros it must be

$$\widetilde{P}'(y) = -Cy^{p-1}(1-y)^{p-1} \quad \text{for some } C. \tag{5.66}$$

The second property comes when we add $\widetilde{P}(y)$ to $\widetilde{P}(1-y)$. The sum equals 2 at both ends and is still flat. Its $2p$ coefficients are uniquely determined — it must be the constant polynomial 2:

$$\widetilde{P}(y) + \widetilde{P}(1-y) \equiv 2. \tag{5.67}$$

At $y=\frac{1}{2}$ this gives $\widetilde{P}(\frac{1}{2}) = 1$. Figure 5.5 shows how $\widetilde{P}(y)$ is odd around its middle value. This "Hermite interpolating polynomial" drops from 2 to 0 with flatness at the ends. Here are the polynomials for $p=2$ and $p=3$:

$$B_2(y) = 1+2y \quad \text{and} \quad \widetilde{P}(y) = 2(1-y)^2(1+2y) = 2-6y^2+4y^3$$
$$B_3(y) = 1+3y+6y^2 \quad \text{and} \quad \widetilde{P}(y) = 2(1-y)^3B_3(y) = 2-20y^3+30y^4-12y^5.$$

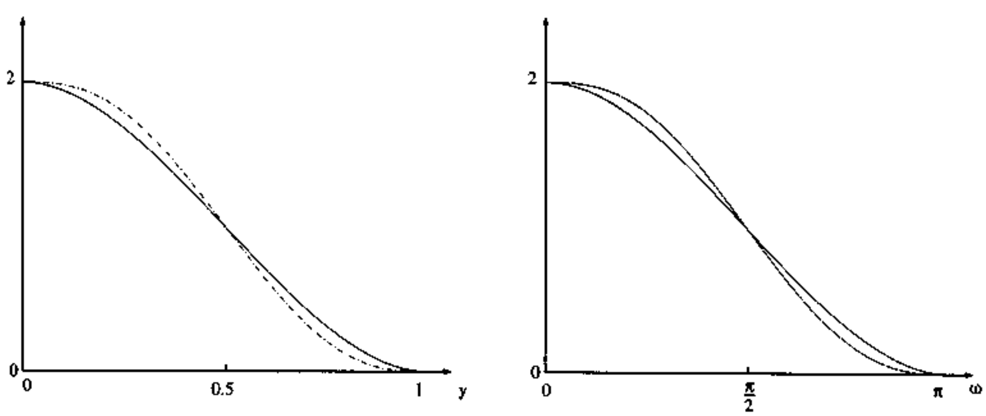


Figure 5.5: $\widetilde{P}(y)$ on the left and $P(\omega)$ on the right, for $p=2$ and $p=3$.

Now we go from ordinary polynomials in y to trigonometric polynomials in ω. The degree stays at $2p-1$. The change that takes $0 \le y \le 1$ into $0 \le \omega \le \pi$ is

$$y = \frac{1-\cos\omega}{2} \quad \text{and} \quad 1-y = \frac{1+\cos\omega}{2}. \tag{5.68}$$

The polynomial $\widetilde{P}(y)$ *becomes our desired* $P(\omega)$. We summarize its properties.

Theorem 5.5 *The polynomial* $2(1-y)^pB_p(y)$ *becomes the halfband response*

$$P(\omega) = 2\left(\frac{1+\cos\omega}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} \left(\frac{1-\cos\omega}{2}\right)^k. \tag{5.69}$$

This satisfies Conditions O and A_p. Its Meyer form, by integrating $P'(\omega)$ and choosing c to give $P(\pi) = 0$, is

$$P(\omega) = 2 - c\int_0^{\omega} (\sin\omega)^{2p-1} d\omega. \tag{5.70}$$

For $p = 1, 2, 3$ the Daubechies and Meyer forms are

$$\begin{aligned} P(\omega) &= \quad 1 + \cos\omega = 2 - \textstyle\int_0^{\omega} \sin\omega\, d\omega \\ P(\omega) &= \quad (1+\cos\omega)^2(1 - \tfrac{1}{2}\cos\omega) = 2 - \tfrac{3}{2}\textstyle\int_0^{\omega}\sin^3\omega\, d\omega \\ P(\omega) &= \quad (1+\cos\omega)^3(1 - \tfrac{9}{8}\cos\omega + \tfrac{3}{8}\cos^2\omega) = 2 - \tfrac{15}{4}\textstyle\int_0^{\omega}\sin^5\omega\, d\omega \end{aligned}$$

Most authors emphasize the Daubechies form, with its highly visible factor $(1 + \cos\omega)^p$. That immediately ensures a pth order zero for the factors at $\omega = \pi$. Spectral factorization is speeded up, because only a lower-degree polynomial remains. It may not be so clear that (5.69) is a half-band filter. The even powers like $\cos^2\omega$ and $\cos^4\omega$ must disappear and they do. In the explicit formula for $p = 2$, multiplication produces $P(\omega) = 1 + \frac{3}{2}\cos\omega - \frac{1}{2}\cos^3\omega$.

The halfband property is $P(\omega) + P(\omega + \pi) \equiv 2$. This addition cancels the odd powers of $\cos\omega$, and the even powers are not present (except the constant term 1). This identity follows immediately from (5.67) because $1 - y = \frac{1+\cos\omega}{2} = \frac{1-\cos(\omega+\pi)}{2}$:

$$\widetilde{P}(y) + \widetilde{P}(1-y) \equiv 2 \text{ becomes } P(\omega) + P(\omega+\pi) \equiv 2. \tag{5.71}$$

The reader recognizes this "Condition O" as $|C(\omega)|^2 + |C(\omega+\pi)|^2 = 2$.

The halfband property is immediate in the Meyer form, with absolutely no calculations. Replace y by $(1 - \cos\omega)/2$ in (5.66) to find $P'(\omega)\, d\omega$:

$$-Cy^{p-1}(1-y)^{p-1}dy = -C\Big(\frac{1-\cos\omega}{2}\Big)^{p-1}\Big(\frac{1+\cos\omega}{2}\Big)^{p-1}\frac{\sin\omega}{2}\, d\omega. \tag{5.72}$$

This is $-c(1-\cos^2\omega)^{p-1}\sin\omega\, d\omega$, which is also $-c(\sin\omega)^{2p-1}\, d\omega$. Its integral is

$$-c\int \left(1-\cos^2\omega\right)^{p-1}\sin\omega\, d\omega = \text{odd powers of } \cos\omega.$$

The only even frequency is a constant of integration. The filter is halfband.

The flatness condition requires first of all that $P(\pi) = 0$. The constant c makes this true. The derivative $P'(\omega) = -c(\sin\omega)^{2p-1}$ has a zero of order $2p - 1$ at $\omega = \pi$. Then P itself has a zero of order $2p$. Its factor C has a zero of order p. Condition A_p is satisfied and Meyer's formula is confirmed.

Note that $P(\omega)$ *decreases monotonically* from $P(0) = 2$ to $P(\pi) = 0$. Its derivative $-c(\sin\omega)^{2p-1}$ is everywhere negative between 0 and π. There are no ripples in Figure 5.5. Therefore $P(\omega) \geq 0$ for all ω, and a factorization into $|C(\omega)|^2$ is assured.

The transition from passband (low frequencies) to stopband (high frequencies) becomes steeper and sharper as p increases. The slope at the midpoint $\omega = \frac{\pi}{2}$ is $-c\left(\sin\frac{\pi}{2}\right)^{2p-1}$, which is $-c$. We will show that c increases asymptotically like $\sqrt{p}$ as $p \to \infty$. Thus the transition band has width of order $1/\sqrt{p}$.

The Halfband Filter $P(z)$

Now we change from y and ω to the complex variable z. This will produce the filter coefficients in $P(z)$. That polynomial will be halfband and centered. The shifted polynomial $P_0(z) = z^{-N}P(z) = z^{1-2p}P(z)$ will be halfband and causal. The change of variables comes from $z = e^{i\omega}$:

$$\frac{z+z^{-1}}{2} = \cos\omega = 1-2y. \tag{5.73}$$

Thus $y = 0$ and $\omega = 0$ give $z = 1$. Similarly $y = 1$ and $\omega = \pi$ give $z = -1$.

Notice that the midpoints $y = \frac{1}{2}$ and $\omega = \frac{\pi}{2}$ give $z = \pm i$. *There are two z's for each y*, from $z+z^{-1} = 2-4y$. (This is a quadratic equation for z.) One z is inside the unit circle, the other is $1/z$ outside. This "Joukowski transformation" is also central in fluid flow. The endpoints $z = 1$ and $z = -1$ are really double roots of $z+z^{-1} = 2$ and $z+z^{-1} = -2$.

The change of variable gives $1-y$ and y in factored form:

$$1-y = \frac{1+\cos\omega}{2} = \left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right) \text{ and } y = \frac{1-\cos\omega}{2} = \left(\frac{1-z}{2}\right)\left(\frac{1-z^{-1}}{2}\right). \tag{5.74}$$

Substituting in $\widetilde{P}(y)$, the maxflat filter in the z-domain becomes $P(z)$:

$$P(z) = 2\left(\frac{1+z}{2}\right)^p\left(\frac{1+z^{-1}}{2}\right)^p \sum_{k=0}^{p-1}\binom{p+k-1}{k}\left(\frac{1-z}{2}\right)^k\left(\frac{1-z^{-1}}{2}\right)^k. \tag{5.75}$$

This factors into $P(z) = C(z)C(z^{-1})$ when $P(\omega)$ factors into $|C(\omega)|^2$. The p zeros at $y = 1$ and $\omega = \pi$ are now $2p$ zeros at $z = -1$. Half of them go into $C(z)$. The $p-1$ complex zeros of the other factor $B_p(y)$ become $2p-2$ zeros of $P(z)$. Half of those (the $p-1$ zeros inside the circle $|z| = 1$, if we want minimum phase) also go into $C(z)$. So the spectral factor $C(z)$ can be computed in two steps:

1. Find the $p-1$ zeros of $B_p(y)$ and the $p-1$ corresponding z's with $|z| < 1$.

2. Include p zeros at $z = -1$. Then $C(z)$ has these $2p-1$ zeros.

Example. $p = 2$ leading to Daubechies D_4 from $B_2(y) = 1+2y$, which is $\frac{1}{2}(-z+4-z^{-1})$.

The zero is at $y = -\frac{1}{2}$. Therefore $z+z^{-1} = 4$. This quadratic equation has roots $z = 2\pm\sqrt{3}$. Then the $2p-1$ roots of $C(z)$ are $-1, -1, 2-\sqrt{3}$. The coefficients of D_4 are approximately 0.4830, 0.8365, 0.2241, and −0.1294:

$$\begin{aligned} C(z) &= \alpha(1+z^{-1})^2(1-(2-\sqrt{3})z^{-1}) \\ &= \left[(1+\sqrt{3}) + (3+\sqrt{3})z^{-1} + (3-\sqrt{3})z^{-2} + (1-\sqrt{3})z^{-3}\right] / 4\sqrt{2}. \end{aligned}$$

Example. $p = 70$ leading to Daubechies D_{140} from $B_{70}(y)$.

From $p = 2$ to $p = 70$ is quite a jump! Figure 5.6 displays the 69 zeros of $B_{70}(y)$. They are close to a limiting curve in the complex y-plane. The equation $|4y(1-y)| = 1$ of that curve is discussed below. In the z-plane the limiting curve is moon-shaped, with the beautiful formula $|z-z^{-1}| = 2$. ***It consists of two circles!*** The roots of the minimum phase factor $C(z)$ are close

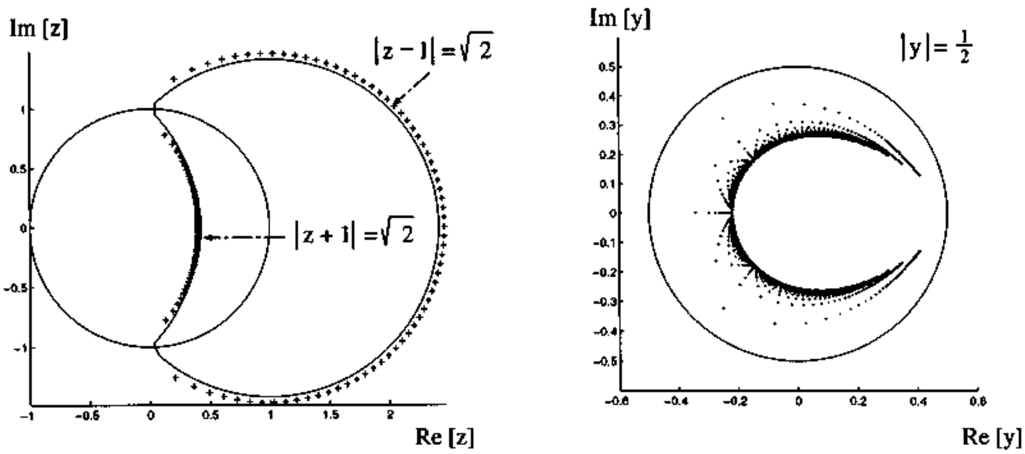


Figure 5.6: The 138 zeros of $P(z)$ and the zeros of $B_p(y)$ up to $p = 60$.

to the inner circle. To those $p - 1 = 69$ inner zeros we add $p = 70$ zeros at $z = -1$. Then the spectral factor of $P(z)$ has 139 roots, and the constant β makes $C(1) = \sqrt{2}$:

$$C(z) = \beta \left(\frac{1 + z^{-1}}{2}\right)^{70} \prod_{1}^{69}\left(1 - z^{-1}Z_j\right). \tag{5.76}$$

Same Coefficients in Interpolation

The coefficients in $-\frac{1}{16} + \frac{9}{16}z^{-2} + z^{-3} + \frac{9}{16}z^{-4} - \frac{1}{16}z^{-6}$ appear in many places. This is a typical example, with $p = 2$, of a maxflat halfband filter. Its factor $(1 + z^{-1})^4$ yields $2p = 4$ zeros at $z = -1$. The halfband property means that $\boldsymbol{P}$ is an ***interpolating filter***: $\boldsymbol{Px}$ keeps the even-numbered coefficients of $\boldsymbol{x} = (\ldots, x(0), 0, x(2), 0, \ldots)$. Four zeros at π mean that the four polynomials $1, t, t^2, t^3$ are correctly interpolated in the odd components:

$$\begin{aligned}
\boldsymbol{x} &= (\ldots, 1, 0, 1, 0, 1, \ldots) & &\text{gives} & \boldsymbol{Px} &= (\ldots, 1, 1, 1, 1, 1, \ldots)\\
\boldsymbol{x} &= (\ldots, 0, 0, 2, 0, 4, \ldots) & &\text{gives} & \boldsymbol{Px} &= (\ldots, 0, 1, 2, 3, 4, \ldots)\\
\boldsymbol{x} &= (\ldots, 0, 0, 4, 0, 16, \ldots) & &\text{gives} & \boldsymbol{Px} &= (\ldots, 0, 1, 4, 9, 16, \ldots)\\
\boldsymbol{x} &= (\ldots, 0, 0, 8, 0, 64, \ldots) & &\text{gives} & \boldsymbol{Px} &= (\ldots, 0, 1, 8, 27, 64, \ldots)
\end{aligned}$$

Expressed differently, $\frac{9}{16}(x(1) + x(-1)) - \frac{1}{16}(x(3) + x(-3))$ is fourth-order accurate at the mid-point $t = 0$. This links wavelet theory to *recursive subdivision* (interpolation to create smooth curves). Starting with equally-spaced values $x(2n)$, $\boldsymbol{P}$ puts new values at the halfway points $t = n$. Then $\boldsymbol{P}$ produces new values at $t = \frac{n}{2}$. The established values do not change. The limit is a smooth curve through the original values. The monograph [CDM] develops this important application, with references. The stability of recursive interpolation is controlled by Condition E applied to $\boldsymbol{P}$.

Asymptotics of the Daubechies Filters

Figure 5.6 practically requires us to study the zeros Y and Z as $p \to \infty$. The first steps were taken by [LeKa] and [ShSt]. The truncated binomial series $B_p(y)$ has degree $p-1$. Its $p-1$ zeros yield $2p-2$ zeros in the z-plane, from $Z + Z^{-1} = 2 - 4Y$. The main facts proved so far are:

1. All the zeros have $|Y| \le \frac{1}{2}$ and $\text{Re}\, Z > 0$.
2. In the y-plane, the zeros are all outside the limiting curve $|w| = |4y(1-y)| = 1$.
3. In the z-plane, the zeros are all outside the limiting curve $|z - z^{-1}| = 2$.
4. The zeros are near a uniform distribution along the circle of radius $1 + \log(4\pi p)/2p$ in the w-plane.
5. The far left zero is $Z = i - W/\sqrt{p} - iW^2/2p + O(p^{-3/2})$ where $\text{erf}(W) = 1$.

Note that if z lies on that moon-shaped limiting curve $|z - z^{-1}| = 2$, so do $\bar{z}$ and z^{-1} and $\bar{z}^{-1}$. The complex roots in the z-plane come *four at a time*, for finite p and in the limit $p = \infty$. The moon consists of two circles of radius $\sqrt{2}$ (Problem). The outer circle is $|z - 1| = \sqrt{2}$ with center at 1. The inner circle is $|z + 1| = \sqrt{2}$ with center at -1. The circles meet at $z = \pm i$, and the far left zeros go slowly toward these two points.

Figure 5.6 shows the zeros in the y-plane up to $p = 60$. Their approach to the limiting curve is fascinating. The application of long filters, with large p, is still to be developed. It can use the lattice structure of Section 4.5

Computing the Spectral Factor $C(z)$

For large p, computing the zeros of $B_p(y)$ and the spectral factorization $P(z) = C(z)C(z^{-1})$ are challenging tasks. They are related but not identical. After we find the zeros, the filter coefficients in $C(z)$ are fully determined — but not necessarily in a well-conditioned way. The direct multiplication of 69 or 139 linear factors is not safe. The cepstral method (Section 5.4) is competitive because it goes directly to $C(z)$, without the zeros. It is based on splitting $\log P(z)$ into $\log C(z) + \log C(z^{-1})$. Our experience with the zeros, using MATLAB and also Lang's code from Rice, showed the importance of a simple weighting:

The zeros of $B_p(y)$ were correct up to $p = 34$. The codes fail for $p = 35$.

With weighted variable $4y$, the zeros became correct up to $p = 80$.

The reason for the breakup at $p = 35$ is the wide dynamic range of coefficients in $B_p(y)$. The constant term is 1. The highest term y^{p-1} has coefficient

$$\frac{(2p-2)!}{(p-1)!(p-1)!} \simeq \frac{\sqrt{2\pi(2p-2)}}{2\pi(p-1)} \frac{(2p-2)^{2p-2}}{(p-1)^{2p-2}} = \frac{4^{p-1}}{\sqrt{\pi(p-1)}} \tag{5.77}$$

by Stirling's formula. The leading term 4^{p-1} multiplying y^{p-1} suggests that $4y$ is a better variable than y. This was strongly confirmed by experiment. The recursion

$$b(p) = 1; \quad \text{for } p-1:1:1 \quad b(i) = b(i+1) * (2p-i-1)/4 * (p-i)$$

produces the coefficients in MATLAB order for the command $Y = \textbf{\textit{roots}}\ (b)/4$.

The zeros are needed in a linear phase factorization $P_0(z) = H_0(z)F_0(z)$, which does not come from the cepstral method. In this case all four zeros $z, \bar{z}, z^{-1}, \bar{z}^{-1}$ go into the same factor (two from inside the unit circle, two from outside). One possibility is to put those quartets alternately in analysis and synthesis, H_0 and F_0, to give filters of nearly equal length. Both get p zeros at $\omega = \pi$. Another possibility is for $H_0(z)$ to be very short, like Haar. Then $F_0(z)$ comes from an exact division $P_0(z)/H_0(z)$. No zeros are needed! Experiment will show which linear-phase factors give the best compression of signals and images.

Transition Band for Maxflat Filters

The equiripple filters from the Remez-Parks-McClellan algorithm have a sharp transition from lowpass to highpass. Their transition band has width of order $\frac{1}{N}$. Their slope at the midpoint $\omega = \frac{\pi}{2}$ is of order N. They minimize the maximum error, giving the best pointwise approximation to 1 in the passband and 0 in the stopband. But they only have one zero (at most) at $\omega = \pi$.

The Daubechies filters, with many zeros at π, have no ripples. $P(\omega)$ and $|C(\omega)|$ decrease monotonically between 0 and π. We now show that the slope at $\omega = \frac{\pi}{2}$ is much smaller, of order $\sqrt{N}$ instead of $N = 2p - 1$.

Theorem 5.6 *The maxflat filter has center slope proportional to $\sqrt{N}$. The transition from $P(\omega) = 0.98$ to $P(\omega) = 0.02$ is over an interval of length $4/\sqrt{N}$.*

Proof. The constant c in Meyer's form is fixed by $c\int_0^\pi (\sin\omega)^N d\omega = 2$. This definite integral is known to be a ratio of Gamma functions (which are factorials $\Gamma(n+1) = n!$). We use Stirling's formula to estimate the integral:

$$\sqrt{\pi}\frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N+2}{2})} \simeq \sqrt{\pi}\left(\frac{N-1}{2e}\right)^{\frac{N-1}{2}}\left(\frac{2e}{N}\right)^{\frac{N}{2}} = \sqrt{\frac{2\pi e}{N-1}}\left(1-\frac{1}{N}\right)^{\frac{N}{2}} \simeq \sqrt{\frac{2\pi}{N}}.$$

The slope of $P(\omega)$ at $\omega = \frac{\pi}{2}$ is $-c \simeq -\sqrt{2N/\pi}$ in Meyer's form. The transition bandwidth is therefore $O(1/\sqrt{N})$ and we make this more precise. Between $\frac{\pi}{2} - \frac{\sigma}{\sqrt{N}}$ and $\frac{\pi}{2} + \frac{\sigma}{\sqrt{N}}$ the drop in $P(\omega)$ is the integral of $c(\sin\omega)^N$. Shift by $\frac{\pi}{2}$ to center the integral, replacing $\sin(\frac{\pi}{2} - \omega)$ by $\cos\omega$:

$$\text{drop} = c\int_{-\sigma/\sqrt{N}}^{\sigma/\sqrt{N}} (\cos\omega)^N d\omega \simeq \frac{c}{\sqrt{N}}\int_{-\sigma}^{\sigma}\left(1-\frac{\theta^2}{2N}\right)^N d\theta \simeq \sqrt{\frac{2}{\pi}}\int_{-\sigma}^{\sigma} e^{-\theta^2/2} d\theta. \qquad (5.78)$$

Here $\theta = \omega\sqrt{N}$. Thus 95% of the drop in $P(\omega)$ comes with $\sigma = 2$ (within two standard deviations of the mean, for the normal distribution in statistics). This transition interval has width $\Delta\omega = 4/\sqrt{N}$, as the theorem predicts. That rule was found experimentally by Kaiser and Reed in 1977, at the beginning of the triumph of digital filters.

Problem Set 5.5

1. The halfband filter $P(\omega) = 1 + \sum p(n)e^{-in\omega}$ (odd n only) satisfies $P(\pi) = 0$. Deduce directly that $P(0) = 2$.

2. Find the zeros of $B_3(y)$ and the Daubechies 6-tap filter D_6.

3. The definite integral $\int_0^\pi (\sin\omega)^N \, d\omega$ equals $\frac{1\cdot3\cdot5\cdots N}{2\cdot4\cdots(N-1)}$ for odd N. Express this in factorials and use Stirling's formula to rederive the estimate $\sqrt{2\pi/N}$. Then the slope at the center frequency $\frac{\pi}{2}$ is $O(\sqrt{N})$.

4. The points $z = 1 + \sqrt{2}e^{i\theta}$ are on the circle $|z - 1| = \sqrt{2}$. Substitute for z to show that this circle is one part of our limiting curve:

$$\left|\frac{z - z^{-1}}{2}\right| = \left|\frac{1 + \sqrt{2}e^{-i\theta}}{1 + \sqrt{2}e^{i\theta}}\right| = 1.$$