Chapter 7

Wavelet Theory

This chapter follows through on important questions about scaling functions and wavelets. We study their smoothness, their inner products, their accuracy of approximation, and the number of vanishing moments. Even more basic are the existence and the construction of $\phi(t)$. *Does the dilation equation have a solution $\phi(t)$ with finite energy? Does the cascade algorithm converge to this solution?* We answer those questions in terms of the filter coefficients $h(k)$ (and the answers are not always yes).

Our overall aim is understanding, with a minimum of technical details. We point immediately to two fundamental operators in wavelet theory. They both have a double shift between rows coming from $(\downarrow 2)$:

$$M = (\downarrow 2)\, 2H \quad \text{and} \quad T = (\downarrow 2)\, 2HH^T. \tag{7.1}$$

Apart from its extra factor 2, M is completely familiar: filter by H and then downsample. This is the decimated lowpass analysis filter, with coefficients $h(k)$ adding to 1. In the time domain, those are on the diagonals of H.

The symmetric product HH^T is also a Toeplitz matrix. Its entries are the coefficients in $|H(\omega)|^2$. The rows are double-shifted by $(\downarrow 2)$. In frequency, downsampling produces an aliasing term from modulation by π:

$$\begin{aligned}(Mf)(\omega) &= H(\tfrac{\omega}{2})\, f(\tfrac{\omega}{2}) + H(\tfrac{\omega}{2}+\pi)\, f(\tfrac{\omega}{2}+\pi)\\ (Tf)(\omega) &= |H(\tfrac{\omega}{2})|^2\, f(\tfrac{\omega}{2}) + |H(\tfrac{\omega}{2}+\pi)|^2\, f(\tfrac{\omega}{2}+\pi)\end{aligned} \tag{7.2}$$

The properties of M and T hold the answers to our questions. Iterating the lowpass filter, with subsampling, involves powers of matrices. The convergence depends on the eigenvalues. That will be the message in this chapter: ***Watch the eigenvalues***.

The "transition operator" or "transfer operator" T turns out to be simpler than M — because $|H(\omega)|^2 \geq 0$ and HH^T is symmetric positive definite. T enters when we compute inner products and energies (L^2 norms). After i iterations of M in the cascade algorithm, starting from $\phi^{(0)}(t)$ and reaching $\phi^{(i)}(t)$, the inner products are

$$a^{(i)}(k) = \int_{-\infty}^{\infty} \phi^{(i)}(t)\, \phi^{(i)}(t+k)\, dt. \tag{7.3}$$

The key point will be that $Ta^{(i)}(k)$ gives the inner products $a^{(i+1)}(k)$. The powers of T (and therefore the eigenvalues of T!) decide whether the cascade algorithm converges, when we use the L^2 energy norm. The properties of $\boldsymbol{M}$ give *pointwise* answers, and the properties of $\boldsymbol{T}$ give *mean square* answers.

We can summarize in a few lines approximately what those answers are:

1. Combinations of $\phi(t-k)$ can exactly produce the polynomials $1, t, \ldots, t^{p-1}$ if $\boldsymbol{M}$ has eigenvalues $1, \frac{1}{2}, \ldots, \left(\frac{1}{2}\right)^{p-1}$; $\boldsymbol{T}$ has eigenvalues $1, \frac{1}{2}, \ldots, \left(\frac{1}{2}\right)^{2p-1}$.

2. The wavelets are orthogonal to $1, \ldots, t^{p-1}$ so they have p vanishing moments. The functions $\phi(t-k)$ give pth order approximation from the space V_0.

3. The cascade algorithm converges to $\phi(t)$ in L^2 if the other eigenvalues of $\boldsymbol{T}$ satisfy $|\lambda| < 1$.

4. $\phi(t)$ and $w(t)$ have s derivatives in L^2 if the other eigenvalues of $\boldsymbol{T}$ satisfy $|\lambda| < 4^{-s}$.

In **1**, the special eigenvalues (powers of $\frac{1}{2}$) give a new form of Condition A_p. It is equivalent to p zeros at π, from a factor $(1+z^{-1})^p$ in $H(z)$.

In 3, the requirement $|\lambda| < 1$ on the other eigenvalues will be called **Condition E**. This is new to the book. It gives convergence of the cascade algorithm to a Riesz basis $\{\phi(t-k)\}$. Condition E *is the key to iteration of filters, and thus to wavelets.*

In **4**, the smoothness s is very likely not an integer (but we work with integers for simplicity). The number s_{max} of derivatives of $\phi(t)$ is never greater than the number p of vanishing moments. The smoothness of $\phi(t)$ is important in image processing, and p is more important.

This chapter begins with the *accuracy of approximation*. That has the most direct answer. The approximation order is p when the frequency response $H(\omega)$ has a pth order zero at $\omega = \pi$. *Zeros at π are the heart of wavelet theory!* This multiple zero is produced by the factor $(1+e^{-i\omega})^p$ that appears throughout the design of filters. $\boldsymbol{T}$ is associated with $|H(\omega)|^2$ for which the order of the zero becomes $2p$. This is reflected in the $2p$ special eigenvalues of $\boldsymbol{T}$.

Sections 7.2–7.3 analyze the cascade algorithm $\phi^{(i+1)}(t) = \sum 2h(k)\phi^{(i)}(2t-k)$. Section 7.4 explains splines and spline wavelets. Section 7.5 introduces multiwavelets.

We note here the MATLAB commands to construct $\boldsymbol{M}$ and $\boldsymbol{T}$ from the vector $\boldsymbol{h}$:

```
function M = down(h)
n = length(h); M = zeros(n,n); for i = 1:n, for j = 1:n,
if (0 < 2 * i - j) & (2 * i - j) <= n) M(i,j) = 2 * h(2 * i - j);
end
end, end
autocor = conv(h, fliplr(h)); T = down(autocor)
```

7.1 Accuracy of Approximation

In applications of wavelets, a function $f(t)$ is projected onto a scaling space V_j. This index j gives the time scale $\Delta t = 2^{-j}$ in the calculation. The scaling functions are $2^{j/2}\phi(2^j t)$ and its translates by $k\Delta t$. Those represent one basis for the space V_j. The projection $f_j(t)$ is the piece of $f(t)$ in that subspace, so it is a combination of those basis functions:

$$\text{For each } j, \quad f_j(t) = \sum_{k=-\infty}^{\infty} a_{jk} 2^{j/2}\phi(2^j t - k). \tag{7.4}$$

This is all at level j. In contrast, wavelets come from splitting functions into several scales. ***Multiresolution*** combines the details at levels zero through $j-1$ and the coarse average at level zero. For subspaces this is $V_j = V_0 \bigoplus W_0 \bigoplus \cdots \bigoplus W_{j-1}$. Except for V_0, the basis functions are now wavelets:

$$f_j(t) = \sum_k a_{0k}\, \phi(t-k) + \sum_k b_{0k}\, w(t-k) + \sum_k b_{1k}\, 2^{1/2} w(2t-k) + \cdots \tag{7.5}$$

In practice, the level j is determined by balancing *accuracy* with *cost*. In other words, we balance *distortion* with *rate*. The cost and the bit rate are approximately doubled between one level and the next — there are twice as many basis functions and twice as many coefficients. This section will estimate the improvement in accuracy (the drop in distortion).

The accuracy depends partly on the filter bank coefficients and partly on $f(t)$. A smooth function and a smooth signal will be easier to approximate and send. Part of our goal is to separate the two contributions to the error (the distortion), one part from the properties of $\boldsymbol{h}(k)$ and the other from the properties of $f(t)$. We choose $\boldsymbol{h}(k)$. The application presents us with $f(t)$. A typical form of the error estimate will involve the pth derivative of $f(t)$:

$$\|f(t) - f_j(t)\| \approx C(\Delta t)^p\, \|f^{(p)}(t)\|. \tag{7.6}$$

The constant C and the exponent p depend on our choice of $\boldsymbol{h}(k)$. That determines $\phi(t)$ and the subspaces. The step from $\Delta t = 2^{-j}$ to $\Delta t = 2^{-(j+1)}$ divides the error by about 2^p. Thus the number p is critical, when we reach the level at which this "asymptotic" error estimate is accurate. Usually the constant C is less critical, but for wavelets it is an order of magnitude larger than for splines.

The smoothness or roughness of $f(t)$ may be outside our control. This contributes to the error through the norm $\|f^{(p)}(t)\|$ of the pth derivative. The global error estimate (7.6) can be made *local*, if there are regions where $f^{(p)}(t)$ is small and also regions of sudden change. The error will be small in one region and large in the other — unless we increase j in the region of sudden change. Then we have an *adaptive mesh*.

A major advantage of wavelets over Fourier methods is this possibility of local refinement. This is the multigrid idea for finite differences, and it is a key virtue of finite elements. Adaptivity holds also for wavelets — mesh refinement is relatively convenient. (The refinement is usually by factors of 2. Irregular meshes are anathema to Fourier.) We add the word "relatively" because adaptivity has an overhead that practitioners have come to respect.

The error estimate (7.6) is completely typical of numerical analysis. It appears in finite elements, where Δt becomes the element size [SF]. It appears in difference methods for initial-value problems ($p = 1$ for Euler's method, $p = 2$ for centered leapfrog, $p = 4$ for Runge-Kutta, etc.). The form of (7.6) is already set by the basic problem of numerical integration:

$$\int_0^1 f(t)dt - \sum c_k f(t_k) \approx \begin{cases} (\Delta t)^1: & \text{rectangle rule} \\ (\Delta t)^2: & \text{midpoint or trapezoid rule} \\ (\Delta t)^4: & \text{Simpson's rule}\ldots \end{cases} \tag{7.7}$$

The exponent is p, the scale length is Δt, and the symbol $\approx$ hides a constant C and a factor $\|f^{(p)}(t)\|$. All these examples, which extend to other numerical approximations too, have the same form because they are based on the same idea.

That key idea is: ***Watch the polynomials***. The rectangular rule is exact for $f(t)$ = constant. The midpoint rule is exact for $f(t)$ = linear. Simpson's rule is exact for $f(t)$ = cubic polynomial.

The exponent p is the degree of the first polynomial that gives an error. Locally, every smooth function looks like a polynomial. This is the essential idea of the Taylor series (and of calculus!). The lowest degree term $C(\Delta t)^p f^{(p)}(t)$ in the error will dominate. We determine the order of accuracy p by computing with polynomials.

An important point is developed at the end of this section. In digital signal processing, the input is not a function $f(t)$. It is a discrete vector $x(n)$. This vector can come from sampling $f(t)$. But if you use the sampled values as the coefficients a_{jk} that enter the filter bank, you are doing violence to the projection. The theory requires a preprocessing step, to transform sampled values to wavelet coefficients. Then postprocessing converts coefficients back to function values. These two steps can be approximated; they should not be ignored.

Note. The constant C is much smaller for splines than for other well-known scaling functions. Compare these asymptotic constants for $p = 2, 3, \ldots, 9$:

Splines	**0.07**	**0.03**	**0.02**	**0.02**	**0.02**	**0.02**	**0.02**	**0.03**
Daubechies	**0.22**	**0.30**	**0.56**	**1.3**	**3.8**	**13.0**	**49.0**	**216.0**

The leading error term is $C(\Delta t)^p f^{(p)}(t)/p!$ with these constants [SwPi,Unser]. This error is $f(t)$ minus its projection onto V_j as $\Delta t = 2^{-j} \to 0$. The growth in the Daubechies constants means roughly that approximation at scale j is only as close as splines at the coarser scale $j-1$ (*half the resolution and a fraction of the work*). There is a price after all for the irregularity of $\phi(t) = D_{2p}(t)$, even though it can reproduce polynomials and achieve the exponent in $(\Delta t)^p$.

Determination of the Accuracy p

Before developing the theory, we compute the number p. The approximation is by translates of $\phi(t)$ and $w(t)$, which can be difficult and intricate functions. *The computation of p always goes back to the lowpass coefficients* $\boldsymbol{h}(k)$. We can determine p from the $\boldsymbol{h}(k)$, or from $H(\omega)$. What we cannot do, and wish to avoid, is the exact projection in continuous time based on $\phi(t)$.

The test for accuracy p is Condition A_p. We recognize it in the time domain, the frequency domain, and now also in the eigenvalue domain.

Theorem 7.1 *The accuracy is p if the lowpass filter coefficients* $\boldsymbol{h}(k)$ *satisfy these three equivalent forms of Condition* A_p*:*

1. *p sum rules on the coefficients:* $\sum_{n=0}^{N} (-1)^n n^j \boldsymbol{h}(n) = 0$ *for* $j = 0, 1, \ldots, p-1$.

2. *p zeros at* π*:* $H(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^p Q(\omega)$ *and* $H(z) = (\frac{1+z^{-1}}{2})^p Q(z)$.

3. *p eigenvalues* $1, \frac{1}{2}, \ldots, \left(\frac{1}{2}\right)^{p-1}$ *of the matrix* $\boldsymbol{M} = (\downarrow 2)2\boldsymbol{H} = \{2\boldsymbol{h}(2i-j)\}$:

$$\boldsymbol{M}\Phi^{(j)} = \left(\tfrac{1}{2}\right)^j \Phi^{(j)} \quad \textit{for} \quad j = 0, 1, \ldots, p-1. \tag{7.8}$$

Proof. The equivalence of **1** and **2** is straightforward. Substituting $\omega = \pi$ in the frequency response yields the alternating sum of coefficients:

$$\sum \boldsymbol{h}(n)e^{-in\pi} = \boldsymbol{h}(0) - \boldsymbol{h}(1) + \boldsymbol{h}(2) - \cdots \tag{7.9}$$

Therefore $H = 0$ at $\omega = \pi$ when the first sum rule holds. For the next rule, when $j = 1$, the derivative of $h(n)e^{-in\omega}$ brings down a factor $-in$:

$$\sum h(n)(-in)e^{-in\pi} = -i(0h(0) - 1h(1) + 2h(2) - \cdots). \tag{7.10}$$

Then $H' = 0$ at $\omega = \pi$ is equivalent to the second sum rule. A similar reasoning applies for higher order zeros and higher sum rules. A p-fold zero at π is equivalent to p sum rules. The notation has assumed an FIR filter but it could be IIR. We turn now to statement **3** about eigenvalues, and we start with examples.

Suppose $H(\omega)$ is the pth power of $\frac{1}{2}(1 + e^{-i\omega})$. It has p zeros at π. The other factor in this $H(\omega)$ is simply $Q = 1$. In that particular case ***all*** the eigenvalues are powers of $\frac{1}{2}$. These examples for $p = 2, 3, 4$ come from double shifts of $1, 2, 1$ and $1, 3, 3, 1$ and $1, 4, 6, 4, 1$:

$$\boldsymbol{m}_2 = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad \boldsymbol{m}_3 = \tfrac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \qquad \boldsymbol{m}_4 = \tfrac{1}{8}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 1 & 4 & 6 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\lambda = 1, \tfrac{1}{2} \qquad \lambda = 1, \tfrac{1}{2}, \tfrac{1}{4} \qquad \lambda = 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}$$

The N by N matrix $\boldsymbol{m}$ has entries $2h(2i-j)$ for $i, j = 0, \ldots, N-1$. We use the letter $\boldsymbol{m}$, because this is a submatrix of $\boldsymbol{M}$. The eigenvalues of $\boldsymbol{m}$ are also eigenvalues of the infinite matrix $\boldsymbol{M}$, with the eigenvectors extended in both directions by zeros. The submatrix $\boldsymbol{m}$ is called $\boldsymbol{m}(0)$ in Section 6.3. The other N by N submatrix is $\boldsymbol{m}(1) = 2h(2i - j)$ for $i, j = 1, \ldots, N$. This matrix $\boldsymbol{m}(1)$ also has the special eigenvalues $(\frac{1}{2})^k$.

To establish that $\boldsymbol{m}$ has these eigenvalues, we increase the number of "zeros at π" one step at a time. At each step, we watch the eigenvectors and eigenvalues. It will be very convenient to work in the z-domain, where each additional zero at $z = -1$ (which is $\omega = \pi$) comes from another factor $(\frac{1+z^{-1}}{2})$. The new eigenvalues of $\boldsymbol{m}$, when $H(z)$ has that extra factor, are ***half*** the old eigenvalues. There is also one new eigenvalue at $\lambda = 1$. Theorem 7.2 describes the new eigenvalues, and its proof completes Theorem 7.1.

Theorem 7.2 *When $H(z)$ is multiplied by $(\frac{1+z^{-1}}{2})$, $\boldsymbol{m}_{\text{new}}$ is one size larger:*

(a) *The eigenvalues λ_{new} are $\frac{1}{2}\lambda_{\text{old}}$. There is an extra eigenvalue $\lambda_{\text{new}} = 1$.*

(b) *The eigenvectors $\boldsymbol{x}_{\text{new}}$ are the differences of the eigenvectors $\boldsymbol{x}_{\text{old}}$:*

$$\boldsymbol{x}_{\text{new}}(k) = \boldsymbol{x}_{\text{old}}(k) - \boldsymbol{x}_{\text{old}}(k-1) \;\; \textit{and} \;\; X_{\text{new}}(z) = (1 - z^{-1})X_{\text{old}}(z). \tag{7.11}$$

The new $\lambda = 1$ has left eigenvector $\boldsymbol{e}_0 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$. The right eigenvector for $\lambda = 1$ gives the values $\phi_{\text{new}}(n)$ of the scaling function at the integers.

Proof. We will write the eigenvalue equation $\boldsymbol{m}_{\text{old}}\boldsymbol{x}_{\text{old}} = \lambda_{\text{old}}\boldsymbol{x}_{\text{old}}$ in the z-domain. Then multiplication by $(\frac{1+z^{-1}}{2})$ gives the corresponding equation for $\boldsymbol{m}_{\text{new}}$. Notice the aliasing term from $(\downarrow 2)$:

$$\boldsymbol{m}\boldsymbol{x} = \lambda\boldsymbol{x} \;\text{ is }\; H(z)X(z) + H(-z)X(-z) = \lambda X(z^2). \tag{7.12}$$

This is for X_{old}. Now give $H(z)$ the extra $(\frac{1+z^{-1}}{2})$ and $X(z)$ the factor $(1 - z^{-1})$:

$$\left(\frac{1+z^{-1}}{2}\right)H(z)(1-z^{-1})X(z) + \left(\frac{1-z^{-1}}{2}\right)H(-z)(1+z^{-1})X(-z) = \tfrac{1}{2}\lambda(1-z^{-2})X(z^2). \tag{7.13}$$

This is the eigenvalue equation $m_{\text{new}}x_{\text{new}} = \lambda_{\text{new}}x_{\text{new}}$. The eigenvalue is multiplied by $\frac{1}{2}$ and the eigenvectors obey (7.11). The whole proof is in $(\frac{1+z^{-1}}{2})(1-z) = \frac{1}{2}(1-z^{-2})$. In the examples m_2, m_3, m_4 above, the eigenvectors for $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ are in the columns of these matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 1 & -2 \\ \frac{1}{2} & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{6} & \frac{1}{2} & 1 & -3 \\ \frac{4}{6} & 0 & -2 & 3 \\ \frac{1}{6} & -\frac{1}{2} & 1 & -1 \end{bmatrix}$$

Take differences of the eigenvectors $0, 1, 0, \ldots$ and $1, -1, 0, \ldots$ in the 2×2 matrix. Those differences $0, 1, -1, \ldots$ and $1, -2, 1, \ldots$ are in the second matrix. They are eigenvectors of m_3 for $\lambda = \frac{1}{2}$ and $\frac{1}{4}$. The new eigenvector for $\lambda = 1$ gives the values $\phi(1) = \phi(2) = \frac{1}{2}$ of the scaling function for $H(z) = (\frac{1+z^{-1}}{2})^3$. Section 6.3 explained how the dilation equation at the integers is exactly $m\Phi = \Phi$ (with $\lambda = 1$). In this example $\phi(t)$ is a *spline* — linear, quadratic, and cubic for $p = 2, 3, 4$.

The 4×4 matrix m_4 comes from $H(z) = (\frac{1+z^{-1}}{2})^4$. Its eigenvectors in the last three columns are differences of the columns in the 3×3 matrix. The first column, for $\lambda = 1$, holds the values $\frac{1}{6}, \frac{4}{6}, \frac{1}{6}$ of the cubic spline at the integers. Section 7.4 develops the special properties of splines.

Also of importance are the ***left eigenvectors***. For the special eigenvalues $1, \frac{1}{2}, \frac{1}{4}, \cdots$ those eigenvectors are "discrete polynomials". This means that the left eigenvector for $\lambda = 2^{-k}$ is a combination of the row vectors $e_0, e_1, \ldots, e_k$:

$$e_0 = [1 \ 1 \ \cdots \ 1]\,, e_1 = [0 \ 1 \ \cdots \ N-1]\,, e_k = \left[0^k \ 1^k \ \cdots \ (N-1)^k\right].$$

The all-ones vector satisfies $e_0 = me_0$, as we have seen before. It is the left eigenvector y_0 of m corresponding to $\lambda = 1$:

$$[1 \ 1 \ 1 \ \cdots \ 1] \begin{bmatrix} 2h(0) & & & \\ 2h(2) & 2h(1) & 2h(0) & \\ & 2h(3) & 2h(2) & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} = [1 \ 1 \ 1 \ \cdots \ 1]\,. \tag{7.14}$$

This just says that the even sum $\sum 2h(2k)$ and the odd sum $\sum 2h(2k+1)$ are equal to 1. That comes from adding and subtracting the first sum rule $h(0) - h(1) + \cdots = 0$ and the lowpass rule $h(0) + h(1) + \cdots = 1$.

When we multiply $H(z)$ by $(\frac{1+z^{-1}}{2})$, the other left eigenvectors of m_{new} come from the left eigenvectors of m_{old}. Where the right eigenvectors take differences of x_{old}, the left eigenvectors take ***sums***. Summing increases the polynomial degree by 1. In the z-domain, this sum corresponds to multiplication by $\frac{1}{1-z} = \frac{z^{-1}}{z^{-1}-1}$:

Theorem 7.3 *The left eigenvector for $\lambda = 1$ is always e_0. The other left eigenvectors in $y_{\text{new}}m_{\text{new}} = \lambda_{\text{new}}y_{\text{new}}$ come from y_{old} by summing and adding Ce_0:*

$$Y_{\text{new}}(z) = \frac{1}{1-z}Y_{\text{old}}(z) + CE_0(z). \tag{7.15}$$

Proof. Left eigenvectors of m are right eigenvectors of m^T. We transpose the operations of $(\downarrow 2)$ and multiplication by $2H(z)$. The transposes are $(\uparrow 2)$ and multiplication by $2H(z^{-1})$. The eigenvalue equation $m^T y^T = \lambda y^T$ has z^2 because of $(\uparrow 2)$:

$$2H(z^{-1})Y_{\text{old}}(z^2) = \lambda Y_{\text{old}}(z). \qquad (7.16)$$

Now give $H(z^{-1})$ the extra factor $(\frac{1+z}{2})$, and divide $Y(z)$ by $1 - z$:

$$\left(\frac{1+z}{2}\right)H(z^{-1})\frac{1}{1-z^{-2}}Y(z^2) = \tfrac{1}{2}\lambda\frac{1}{1-z}Y(z). \qquad (7.17)$$

This is $m^T_{\text{new}} y^T_{\text{new}} = \lambda_{\text{new}} y^T_{\text{new}}$, which completes the proof. The eigenvalue is multiplied by $\frac{1}{2}$ (of course, since m^T has the same eigenvalues as m). The summation in the left eigenvectors (*row vectors*) can be seen in the same three examples m_2, m_3, m_4:

$$\begin{matrix}\lambda = 1\\ \lambda = \frac{1}{2}\end{matrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{matrix}\lambda = 1\\ \frac{1}{2}\\ \frac{1}{4}\end{matrix}\begin{bmatrix} 1 & 1 & 1 \\ \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{matrix}\lambda = 1\\ \frac{1}{2}\\ \frac{1}{4}\\ \frac{1}{8}\end{matrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ \frac{11}{6} & \frac{2}{6} & -\frac{1}{6} & \frac{2}{6} \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

These left eigenvector matrices are the inverses of the previous right eigenvector matrices! The left eigenvectors are always biorthogonal to the right eigenvectors. The diagonalization by $S^{-1}MS$ has the right eigenvectors in the columns of S and the left eigenvectors in the rows of S^{-1}.

The left eigenvector for $\lambda = 1$ is always e_0, the row of ones. The other left eigenvectors of m_3 come from *minus* the sum of the m_2 eigenvectors, *plus constant*:

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The vector after the arrow has a minus sign and delay. This is because $\frac{1}{1-z}$ equals $-z^{-1}$ times the summing operator $1 + z^{-1} + z^{-2} + \cdots$.

Extension to Infinite Matrices

For the infinite matrix M, these left eigenvectors are ***not*** extended by zeros. The finite eigenvector e_0 becomes an infinite all-ones eigenvector. The linear vector $e_1 = [0 \;\; 1 \;\; \cdots \;\; N-1]$ becomes infinite too: $e_1(n) = n$ for all n. All "polynomial vectors" e_k extend as polynomials. Then the combination of the e's that gives the left eigenvector of m also gives the extended left eigenvector of the infinite M.

The eigenvector $[\; \frac{3}{2} \;\; \frac{1}{2} \;\; -\frac{1}{2} \;]$ of m_3 extends to a linear left eigenvector of M_3:

$$\begin{bmatrix} \cdots & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & \cdots \end{bmatrix}\frac{1}{4}\begin{bmatrix} \cdots & 0 & 0 & 0 & 0 & \\ & 3 & 1 & 0 & 0 & \\ & 1 & 3 & 3 & 1 & \\ & 0 & 0 & 1 & 3 & \cdots \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} \cdots & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & \cdots \end{bmatrix}.$$

Being linear, it has $\lambda = \frac{1}{2}$. The other eigenvector $[\ 1 \quad 0 \quad 0\ ]$ of m_3 extends to a "***quadratic***" left eigenvector (for $\lambda = \frac{1}{4}$) of M_3:

$$[\ 1\ \ 0\ \ 0\ ] \ = \ [\ 1\ \ 1\ \ 1\ ] - \tfrac{3}{2}[\ 0\ \ 1\ \ 2\ ] + \tfrac{1}{2}[\ 0\ \ 1\ \ 4\ ] \ = e_0 - \tfrac{3}{2}e_1 + \tfrac{1}{2}e_2.$$

The nth component of the eigenvector will be $1 - \frac{3}{2}n + \frac{1}{2}n^2$. Now we explain why these left eigenvectors are important in wavelet theory.

Theorem 7.4 *The left eigenvector in $y_k M = (\frac{1}{2})^k y_k$ gives the combination of scaling functions $\phi(t+n)$ that equals t^k:*

$$\sum y_k(n)\phi(t+n) = t^k \quad \textit{for} \quad k = 0, 1, \ldots, p-1. \tag{7.18}$$

Thus the space V_0 spanned by $\{\phi(t+n)\}$ contains all polynomials of degree less than p.

Proof. We are to show that the inner product $G(t) = y_k \Phi_\infty(t) = \sum y_k(n)\phi(t+n)$ equals a multiple of t^k. Here y_k is a left eigenvector of M and $\Phi_\infty(t) = M\Phi_\infty(2t)$ solves the dilation equation. Put those two facts together:

$$y_k\,\Phi_\infty(t) = y_k\,M\,\Phi_\infty(2t) = \left(\tfrac{1}{2}\right)^k y_k\,\Phi_\infty(2t). \tag{7.19}$$

The left side is $G(t)$ and the right side is $(\frac{1}{2})^k G(2t)$. Since those are equal, $G(t)$ is a multiple of t^k. That is what we really wanted to prove. More details are in [HeStSt].

The all-ones eigenvector e_0 says that $\sum \Phi(t+n) = 1$. This constant function assures at least $p = 1$. Since the wavelet is orthogonal to 1, we have $\int w(t)\,dt = 0$ — the first vanishing moment. Hopefully there are more, and $p > 1$.

For M_2, it is no surprise that 1 and t can be produced from translates of the hat function. More important is that 1 and t can be produced from the Daubechies scaling function $\phi(t) = D_4(t)$. This is a typical case in which $H(z)$ has an extra factor $\frac{1}{2}\left[1 + \sqrt{3} + (1-\sqrt{3})z^{-1}\right]$ for double-shift orthogonality. The matrix m is 3 by 3 but only two eigenvalues 1 and $\frac{1}{2}$ are special (with their constant and linear left eigenvectors y_0 and y_1):

$$\frac{1}{4}\begin{bmatrix} 1+\sqrt{3} & & \\ 3-\sqrt{3} & 3+\sqrt{3} & 1+\sqrt{3} \\ & 1-\sqrt{3} & 3-\sqrt{3} \end{bmatrix} \quad \text{has} \quad \begin{array}{ll} \lambda = 1 & y_0 = [1\ \ 1\ \ 1] \\ \lambda = \frac{1}{2} & y_1 = [3-\sqrt{3}\ \ \ 1-\sqrt{3}\ \ \ -1-\sqrt{3}]/2 \\ \lambda = \frac{1+\sqrt{3}}{4} & y_2 = [1\ \ 0\ \ 0] \end{array}$$

The sum $\sum \phi(t-n)$ is identically 1. The combination $\sum y_1(n)\phi(t+n)$ equals a multiple of t. Thus the Daubechies space V_0 contains 1 and t. Those are orthogonal to the wavelets in W_0. This orthogonality $\int w(t)\,dt = 0$ and $\int t\,w(t)\,dt = 0$ says that the Daubechies wavelet has two vanishing moments.

Corollary (7.4) *When $H(\omega)$ has p zeros at π, the wavelets orthogonal to $\phi(t-n)$ have p vanishing moments. Those are the synthesis wavelets $\widetilde{w}(t)$:*

$$\int_{-\infty}^{\infty} \widetilde{w}(t)\,dt = 0, \quad \int_{-\infty}^{\infty} t\,\widetilde{w}(t)\,dt = 0, \quad \ldots, \quad \int_{-\infty}^{\infty} t^{p-1}\widetilde{w}(t)\,dt = 0. \tag{7.20}$$

Reason: $1, \dots, t^{p-1}$ are combinations of $\phi(t-n)$. Orthogonality to these polynomials means p vanishing moments for the wavelets.

Remember that V_0 is orthogonal to $\widetilde{W}_0$ rather than W_0. Thus it is $\widetilde{w}(t)$, not $w(t)$, that has p vanishing moments! For an orthogonal example like a Daubechies filter, $\widetilde{W}_0 = W_0$ and $\widetilde{w}(t) = w(t)$. In the biorthogonal case, the *analysis* wavelet $w(t)$ has $\widetilde{p}$ vanishing moments when the lowpass *synthesis* filter has $\widetilde{p}$ zeros at π.

Note. The polynomials $1, \dots, t^{p-1}$ are not actually in the subspace V_0. They are indeed combinations of the translates of $\phi(t)$. But polynomials have infinite energy, $\int_{-\infty}^{\infty}(t^j)^2 dt = \infty$. If V_0 is defined as a subspace of L^2, it cannot contain polynomials.

This point is only formal, not essential. The eigenvectors y have infinitely many nonzero components, which multiply all translates of $\phi(t)$. This maintains the polynomial for all time. Figure 7.1 shows how a finite combination maintains the polynomials 1 and t for finite time. This figure uses the Daubechies scaling function $\phi = D_4$ which has $p = 2$.

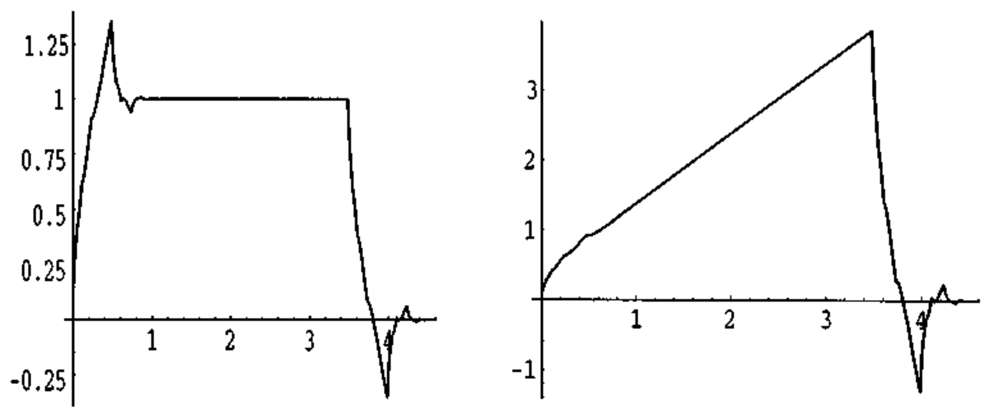

Figure 7.1: A combination of $D_4(t+n)$ can exactly reproduce 1 and t on any interval.

Approximation by Functions in V_j

In continuous time, vectors and matrices are replaced by functions of t. The lowpass filter coefficients $h(k)$ have done their part. By iteration they led to $\phi(t)$. Its translates can reproduce $1, \dots, t^{p-1}$ —and thus all polynomials of degree less than p. Its support interval is $[0, N]$ and we can estimate its smoothness. The theory in continuous time is now a job for "*harmonic analysis*".

Harmonic analysis is the study of function spaces and transforms. It takes its name from the all-important Fourier example, which analyzes $f(t)$ into a sum of harmonics $e^{i\omega t}$. The key problem is *to connect the properties of $f(t)$ with the properties of the transform* (especially the size of the Fourier coefficients). Because $\sin \omega t$ and $\cos \omega t$ and $e^{i\omega t}$ have nonlocal support, cancellation is crucial. The size of the Fourier coefficients tells a lot, but not everything. Only in the energy norm do we have a perfect match:

$$\text{The energy } \int |f(t)|^2\, dt \text{ equals the energy } \frac{1}{2\pi}\int |\widehat{f}(\omega)|^2\, d\omega.$$

In other L^p norms, and in other function spaces, the magnitude $|\widehat{f}(\omega)|$ does not completely decide whether $f(t)$ belongs to the space. We need the phase, which is more difficult. The theory using magnitudes is important. It can never be complete.

For a local wavelet basis, this situation is radically different. The magnitudes are enough! We can match the space of functions $f(t)$ to a space of wavelet coefficients b_{jk}. For $f(t)$ in L^p the coefficients are in the discrete space ℓ^p:

$$A\int |f(t)|^p\,dt \le \sum_{j,k} |b_{jk}|^p \le B\int |f(t)|^p\,dt. \tag{7.21}$$

In the language of harmonic analysis, wavelets are an ***unconditional basis*** when $p > 1$. The magnitudes $|b_{jk}|$ give sufficient information without phases. For L^2, which is always the simplest and clearest, an unconditional basis is a ***Riesz basis***:

$$A\int \left|\sum a_n\,\phi(t-n)\right|^2 dt \le \sum |a_n|^2 \le B\int \left|\sum a_n\,\phi(t-n)\right|^2 dt.$$

Then the translation invariance of the basis $\phi(t-n)$ yields the requirement (Section 6.5) on $A(\omega) = \sum a(k)\,e^{ik\omega}$ in the frequency domain:

$$0 < A \le A(\omega) = \sum_{-\infty}^{\infty} \left|\widehat{\phi}(\omega + 2\pi k)\right|^2 \le B \quad \text{for all } \omega. \tag{7.22}$$

Exact numbers A and B will come from the components $a(k)$ of the eigenvector $\boldsymbol{a} = \boldsymbol{T}\boldsymbol{a}$. So will a similar inequality for the wavelet basis $w_{jk}(t)$ and the coefficients b_{jk}.

Approximation by Wavelets: Errors $f(t) - f_j(t)$

The number p of zeros at π tells how many basis functions are needed to approximate $f(t)$. The smoother the function, and the higher the order p, the faster the expansion coefficients go to zero and the fewer we need to keep.

We are touching here on the central problem of transform analysis — *to find a convenient basis that yields accurate approximation of the signal with few basis functions.* The best basis depends on the signal (of course). We have to choose a basis for a class of signals. For smooth signals, the Fourier basis is usually satisfactory. Perhaps the essential message of wavelet theory can be captured in a sentence:

For piecewise smooth functions, a wavelet basis is better.

These functions may have jumps. They may be smooth and suddenly rough. A wavelet basis, which is local, can separate those pieces. We keep more coefficients in the rough neighborhoods, by going to a smaller scale 2^{-j}. The mesh adapts to $f(t)$ in a way that Fourier finds difficult.

Here is the fundamental theorem on approximation by scaling functions and/or wavelets. The space of approximating functions is V_j, so the scale is $\Delta t = 2^{-j}$. This space is spanned by the scaling functions $\phi(2^j t - k)$, and it is also spanned by the wavelets $w_{jk}(t)$ at all scales below j. We may choose either basis, since multiresolution says that $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$. The basis is not important at this point, because we are looking for the best function in the space.

Theorem 7.5 *When $H(\omega)$ has p zeros at π, any $f(t)$ with p derivatives is approximated to order $(\Delta t)^p = 2^{-jp}$ by its projection $f_j(t)$ in V_j:*

$$\|f(t) - f_j(t)\| \le C(\Delta t)^p \|f^{(p)}(t)\|$$

$$\|f(t) - \sum_k a_k 2^{j/2}\phi(2^j t - k)\| \le C2^{-jp}\|f^{(p)}(t)\|. \tag{7.23}$$

This follows the expected pattern. For approximation by box functions or Haar wavelets, the error is of order Δt because $p = 1$. *Try this piecewise constant approximation on the linear function $f(t) = t$.* The closest constant on the interval $[0, \Delta t]$ is the halfway choice $a_1 = \frac{\Delta t}{2}$. The error $f(t) - f_0(t)$ on this interval is $t - \frac{\Delta t}{2}$. The maximum error is $\frac{\Delta t}{2}$ at $t = 0$. The L^2 error is

$$\|t - \frac{\Delta t}{2}\| = \left[\frac{1}{\Delta t}\int_0^{\Delta t}\left(t - \frac{\Delta t}{2}\right)^2 dt\right]^{1/2} = \frac{\Delta t}{2\sqrt{3}}. \tag{7.24}$$

The first derivative $f^{(1)}(t)$ on the right side of (7.23) is just 1. The constant is $C = \frac{1}{2}$ in the maximum norm. It is $C = 1/2\sqrt{3}$ if we use the L^2 norm. The important part is the power of Δt. But if a different $\phi(t)$ gives a much larger constant C, that is important too.

The main point is that the basis $\{\phi(t-k)\}$ can locally produce $1, \ldots, t^{p-1}$. In each interval we can "essentially" match the start of the Taylor series. The error is the first Taylor series term we ***cannot*** match. This produces $(\Delta t)^p f^{(p)}(t)$ in the error bound. A detailed proof would lead far into approximation theory. The theorem was known earlier in the particular case of splines. Then it was extended to include ***finite elements*** [SF]. Those are local basis functions — normally piecewise polynomials. When the approximation theorem was first proved, nobody was thinking of wavelets! Who would use irregular functions to approximate smooth functions? It defies common sense, but wavelets come from iterations of simple filters. The computations are quick (if done recursively). The theorem applies because those irregular functions $\phi(t-k)$ can exactly reproduce polynomials.

The requirement on $H(\omega)$ is p zeros at π. It is interesting to note the corresponding requirements on $\widehat{\phi}(\omega)$. These are the so-called Strang-Fix conditions:

$$\widehat{\phi}(\omega) \text{ must have zeros of order } p \text{ at all frequencies } \omega = 2\pi n,\ n \neq 0. \tag{7.25}$$

The connection to zeros of $H(\omega)$ is through the infinite product $\widehat{\phi}(\omega) = \prod_1^\infty H(\omega/2^j)$. At frequency $\omega = 2\pi n$, we write $n = 2^j q$ with q odd. Then the $(j+1)$st factor in the infinite product is $H(2\pi n/2^{j+1}) = H(q\pi)$. By periodicity this is $H(\pi)$. A pth order zero of H at $\omega = \pi$ yields a pth order zero of $\widehat{\phi}$ at $\omega = 2\pi n$. Thus the Strang-Fix condition on $\phi(t)$ becomes Condition A_p on $H(\omega)$.

The goal is always to find conditions on $H(\omega)$ that make $\widehat{\phi}(\omega)$ do what we want. For good approximation, we require zeros at π. This same condition improves the smoothness of $\phi(t)$ and stabilizes the lowpass filter under iteration. It is an open question ***how many*** zeros to ask for. Enough to stabilize the iterations, but not so many that we overconstrain the lowpass filter. Designers are often satisfied with two derivatives for $\phi(t)$, which occurs for $p \approx 4$. Other designers accept a smaller p.

Decay of the Wavelet Coefficients

The order p enters in another way, to improve compression. It allows the wavelet coefficients to do what Fourier coefficients do automatically: *they decrease rapidly for a smooth function.* For Fourier, $f(t)$ has to be smooth everywhere. One small jump, and the coefficients decrease no faster than $\frac{1}{k}$. For wavelets, the slow decrease will only apply around the jump. In smooth regions the coefficients drop off quickly. Multiresolution offers a wavelet basis in which the coefficients are directly linked to local properties of $f(t)$.

Theorem 7.6 *If $f(t)$ has p derivatives, its wavelet coefficients decay like 2^{-jp}:*

$$|b_{jk}| = \left|\int f(t)\, w_{jk}(t)\, dt\right| \le C\, 2^{-jp} \|f^{(p)}(t)\|. \tag{7.26}$$

Proof. We plan to integrate by parts in $\int f(t) w_{jk}(t)\, dt$. That gives the derivative of f and the integral of w. (For biorthogonal wavelets replace w by $\widetilde{w}$.) The first vanishing moment means that the integral of $w(t)$ from $-\infty$ to ∞ is zero. *The indefinite integral has compact support:*

$$I_1(t) = \int_{-\infty}^{t} w(u)\, du \quad \text{is nonzero only on } [0, N].$$

$I_1(t)$ is bounded, with finite energy. It will be a hat function, when $w(t)$ is Haar's up-down square wave. Integrate by parts in (7.26) to produce f' times a rescaled I_1:

$$b_{jk} = 2^{-j} \int_{-\infty}^{\infty} f'(t) 2^{j/2} I_1(2^j t - k) dt = O(2^{-j}). \tag{7.27}$$

The factor 2^{-j} comes because we are integrating $w_{jk}(t)$ instead of $w(t)$:

$$\int_{-\infty}^{t} w_{jk}(t) dt = \int_{-\infty}^{t} 2^{j/2} w(2^j t - k) dt = 2^{-j} \int_{-\infty}^{2^j t - k} w(u)\, du.$$

Now repeat this step p times. Each integration by parts brings an extra derivative of $f(t)$ and an extra integral of $w_{jk}(t)$ — with its factor 2^{-j}. If successive integrals of $w(t)$ are $I_1(t)$ and $I_2(t)$ and finally $I_p(t)$, we end with

$$|b_{jk}| = \left|2^{-jp} \int_{-\infty}^{\infty} f^{(p)}(t) 2^{j/2} I_p(2^j t - k) dt\right| \le C 2^{-jp} \|f^{(p)}(t)\|.$$

Even when $f(t)$ has more derivatives, we cannot continue beyond p. The integral of $I_p(t)$ from $-\infty$ to ∞ is not zero! If it were, p integrations by parts would bring us back to $\int t^p w(t) dt = 0$ — but this pth moment of $w(t)$ does ***not*** vanish. $I_{p+1}(t)$ is a nonzero constant when t is large. Its energy is infinite and (7.26) breaks down for $p+1$. Look again at Haar wavelets with $p = 1$. They have a square wave up and a square wave down, at scale 2^{-j}. The differences $f(t) - f(t - 2^{-j})$ are of order 2^{-j}. Integration gives a direct estimate $|b_{jk}| = O(2^{-j})$ in this particular case. The general method integrates by parts to get the derivative $f'(t)$ times the hat function I_1 times the key factor 2^{-j}.

Sample Values vs. Expansion Coefficients

Start with a function $x(t)$. Its samples $x(n)$ are often the inputs to the filter bank. Is this legal? ***No. It is a wavelet crime.*** Some can't imagine doing it, others can't imagine not doing it. Is this crime convenient? ***Yes.*** We may not know the whole function $x(t)$, it may not be a combination of $\phi(t-k)$, and computing the true coefficients in $\sum a(k)\phi(t-k)$ may take too long. But the crime cannot go unnoticed — we have to discuss it.

When the samples $x(n)$ are direct inputs to the filter bank (at unit scale $j=0$), you effectively assume a particular continuous-time function. The pyramid algorithm (filter and downsample) acts on the numbers $x(n)$ as if they were expansion coefficients of its underlying function $x_s(t)$:

$$\textit{Samples as coefficients:} \quad x_s(t) = \sum x(n)\phi(t-n). \tag{7.28}$$

Does $x_s(t)$ have the correct sample values $x(n)$? This seems a minimum requirement. It holds when $\phi(k) = \delta(k)$. Then the only term in (7.28) at $t=n$ is the correct $x(n)$. A centered hat function has this property but most $\phi(t)$ do not. One possible solution is to adjust the coefficients in $\sum a(k)\phi(t-k)$ to produce the known samples $x(n)$:

$$\textit{Determine } a_{int}(k) \textit{ from} \quad x(n) = \sum a_{int}(k)\phi(k-n). \tag{7.29}$$

This linear system has a constant-diagonal Toeplitz matrix. The n, k entry is $\phi(k-n)$. We are inverting an FIR filter that has the response $\sum \phi(k)e^{-ik\omega}$. Then the underlying function that interpolates the samples $x(n)$ is $x_{int}(t) = \sum a_{int}(k)\phi(t-k)$.

We believe that generally the samples $x(n)$ should be pre-filtered, before they enter the filter bank. Solving (7.29) puts the samples through an IIR filter. A different approach yields an FIR filter, by approximating the "correct" coefficients $a(k)$. Those are inner products of $x(t)$ with the analyzing function $\widetilde{\phi}(t-k)$. The pre-filter replaces this integral by a sum:

$$\text{Replace} \quad a(k) = \int x(t)\widetilde{\phi}(t-k)\,dt \quad \text{by} \quad a_q(k) = \sum x(n)\widetilde{\phi}(n-k). \tag{7.30}$$

In the Daubechies examples, $\widetilde{\phi} = \phi$ from orthogonality and the tilde disappears.

The pre-filter that gives $a_q(k)$ is normally FIR. Except for sinc wavelets and the duals to splines, our basis functions have compact support. The ideal function underlying this pre-filter is $x_q(t) = \sum a_q(k)\phi(t-k)$, a sensible choice.

Important: In continuous time, the synthesis $x(t) = \sum a(k)\phi(t-k)$ is consistent with the analysis $a(k) = \int x(t)\widetilde{\phi}(t-k)\,dt$. In discrete time those are *not consistent*. When t changes to n and the integral changes to a sum, inverse operators do not become inverse matrices (unfortunately). But the discrete approximations are exactly correct for polynomials up to order p or $\widetilde{p}$:

$$\sum_{-\infty}^{\infty} n^r\phi(n) = \int_{-\infty}^{\infty} t^r\phi(t)\,dt \quad \text{for} \quad r < p. \tag{7.31}$$

The right side is the rth derivative of $\widehat{\phi}(\omega) = \int \phi(t)e^{-i\omega t}dt$ at $\omega = 0$, times i^r. For the left side we use Poisson's summation formula (Chapter 2). This gives the same rth derivative at $\omega = 0$ and at points $\omega = 2\pi k$:

$$\sum_{-\infty}^{\infty} n^r\phi(n) = \sum_{-\infty}^{\infty} i^r\widehat{\phi}^{(r)}(2\pi k).$$

But all terms on the right are zero except for $k = 0$, by the Strang-Fix condition (7.25) on the function $\phi(t)$. Thus the equality (7.31) holds for any $\phi(t)$ whose translates can reproduce polynomials to degree p. Similarly $a_q(k) = a(k)$, sum equals integral, when $x(t)$ is a low-degree polynomial.

Summary. We recommend that the samples $x(n)$ be converted to coefficients $a_q(k)$ by (7.30). Those enter the filter bank, not $x(n)$. The output $\widehat{a}(k)$ can be post-filtered to recover sample values.

Other pre-filters are also reasonable. [Flandrin] proposed that the underlying $x(t)$ should be band-limited. The sampling theorem gives $x(t)$ as a sum of $\text{sinc}(t-n)x(n)$. Projecting this band-limited $x(t)$ onto V_0 gives $\sum a_{bl}(k)\phi(t-k)$, and those coefficients a_{bl} can enter the filter bank.

The samples $x(n)$ could be regarded as *averages* instead of point values. Then $x(t)$ is assumed piecewise constant, a combination of box functions $B(t-n)$. Projecting $x(t)$ onto V_0 gives $\sum a_{ave}(k)\phi(t-k)$. The filters can operate on $a_{ave}(k)$.

There is no unique answer to our question of how to process the sample values. We may choose $a_{int}(k)$ or $a_q(k)$ or $a_{bl}(k)$ or $a_{ave}(k)$. We should not send samples automatically through the filter bank.

Problem Set 7.1

1. Find the accuracy p from the sum rules for these filter coefficients:

   (a) $h = (\ \frac{1}{4},\ \ \frac{1}{2},\ \ \frac{1}{4}\ )$

   (b) $h = \frac{1}{16}(\ 1,\ \ 4,\ \ 6,\ \ 4,\ \ 1\ )$

   (c) $h = \frac{1}{8}(\ 1-\sqrt{3},\ \ 3-\sqrt{3},\ \ 3+\sqrt{3},\ \ 1+\sqrt{3}\ )$ = Daubechies in reverse.

   (d) $h = D_6$

   (e) $h = (\ \frac{1}{3},\ \ \frac{1}{3},\ \ \frac{1}{3}\ )$

2. Factor the frequency response for filters (a) to (e) into $H(\omega) = (\frac{1+e^{-i\omega}}{2})^p R(\omega)$.

3. Find the eigenvalues of the N by N matrix m for each of the filters (a) to (e). The number of taps is $N+1$; cases (b) and (d) are less convenient by hand.

4. Which 5 by 5 matrix m_5 comes from $H(z) = (\frac{1+z^{-1}}{2})^5$? Find the right eigenvectors for $\lambda = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ from differences of the eigenvectors of m_4 in the text. Find directly the eigenvector for $\lambda = 1$.

5. Verify that the left eigenvectors given in the text for m_4 are sums of the left eigenvectors for m_3, plus a constant $[c\ \ c\ \ c\ \ c]$. Find the five left eigenvectors of m_5.

6. $M = (\downarrow 2)2H$ transforms $\widehat{f}(\omega)$ into $\widehat{Mf}(\omega) = H\left(\frac{\omega}{2}\right)\widehat{f}\left(\frac{\omega}{2}\right) + H\left(\frac{\omega}{2}+\pi\right)\widehat{f}\left(\frac{\omega}{2}+\pi\right)$. The first sum rule is $H(\pi) = 0$. If $\widehat{f}(0) = 0$, show that $\widehat{Mf}(0) = 0$, and explain what this means in the time domain.

7. The first two sum rules are $H(\pi) = 0$ and $H'(\pi) = 0$. Suppose that $\widehat{f}(0) = \widehat{f}'(0) = 0$. Show that $\widehat{Mf}(0) = (\widehat{Mf})'(0) = 0$.

8. If $w(t)$ has p vanishing moments, show that its Fourier transform has a pth order zero at $\omega = 0$. Then factoring $(i\omega)^p$ from $\widehat{w}$ gives the transform $\widehat{I}_p$ of the p-fold integral of $w(t)$.

9. Find the Fourier transform of the Haar wavelet and factor out $i\omega$ to obtain the transform of $I_1(t)$. Show that $I_1(t)$ is a hat function.

10. The left eigenvector $[\ 1 \quad 0\ ]$ for m_2 extends to $[\ \cdots \quad 2 \quad 1 \quad 0 \quad -1 \quad \cdots\ ]$ for the infinite matrix M_2. Write out $yM_2 = \frac{1}{2}y$ and circle the 2 by 2 subvector and submatrix in the middle. Verify that $\cdots + 2\phi(t-1) + \phi(t) + 0 - \phi(t-1) - \cdots$ equals t, when $\phi(t)$ is the hat function from this filter.

11. How would you compute $a_{ave}(k)$ so that $\sum a_{ave}(k)\phi(t-k)$ is the projection onto V_0 of the piecewise constant $x(t)$ with *average* $x(k)$ over the kth subinterval?

7.2 The Cascade Algorithm for the Dilation Equation

In the theory of wavelets, the two-scale dilation equation $\phi(t) = \sum 2h(k)\phi(2t-k)$ is central. Its solution is the scaling function, which leads to wavelets. The equation arises in the limit of the *cascade algorithm*

$$\phi^{(i+1)}(t) = \sum 2h(k)\phi^{(i)}(2t-k). \tag{7.32}$$

This is an iteration (with rescaling) of the lowpass filter. We find a simple proof of the necessary and sufficient condition on the $h(k)$ for $\phi^{(i)}(t)$ to converge in L^2. The cascade normally begins from $\phi^{(0)}(t)$ = box function. Problem 4 finds all other $\phi^{(0)}(t)$ that yield convergence to $\phi(t)$. Thus there are two conditions for convergence, one on the filter (this is the important one) and a condition on $\phi^{(0)}(t)$.

Our method is to compute the inner products $a^{(i)}(k) = \int \phi^{(i)}(t)\phi^{(i)}(t+k)dt$ at each step of the algorithm. The vectors $a^{(i+1)}$ and $a^{(i)}$ are connected by a *transition matrix* T formed from the $h(k)$. The cascade algorithm for $\phi(t)$ becomes the power method $a^{(i+1)} = Ta^{(i)}$ for the equation $a = Ta$. "**Condition E**" for convergence is that all eigenvalues of T satisfy $|\lambda| < 1$ except for a simple eigenvalue at $\lambda = 1$.

Recall that $\lambda = 1$ is an eigenvalue of $M = (\downarrow 2)2H$ by the *first sum rule*:

$$\sum_{\text{even } k} 2h(k) = \sum_{\text{odd } k} 2h(k) = 1.$$

These even k and odd k appear in separate columns of M. Each column adds to 1. Therefore $\lambda = 1$ is an eigenvalue, and the left eigenvector is $e = [1 \ \ 1 \ \ \cdots \ \ 1]$. This is necessary for convergence, pointwise or L^2, but far from sufficient. It means that $H(\omega) = \sum h(k)e^{-ik\omega}$ has a zero at $\omega = \pi$. Other things being equal, every zero at π gives a boost to convergence. This is a double zero in the nonnegative function $P(\omega) = |H(\omega)|^2$. The key to L^2 convergence is the matrix T associated with $|H(\omega)|^2$ in the same way that M is associated with $H(\omega)$:

$$T = (\downarrow 2)2HH^T.$$

When H and H^T are infinite Toeplitz matrices, M and T are also infinite. They are block Toeplitz matrices, with 1 by 2 blocks because of the operator $(\downarrow 2)$. T is illustrated in equation (7.38) below. The important action is in the finite matrix at the center. The central submatrix of T has order $2N-1$:

$$T_{jk} = 2p(2j-k) \quad \text{if} \quad |H(\omega)|^2 = \sum p(k)e^{-ik\omega}, \quad -N < j,k < N.$$

The columns of T are still even or odd, containing coefficients $p(2n)$ or else $p(2n+1)$. Those columns add to 1, since $|H(\omega)|^2$ has a zero at π. The all-ones vector e is still a left eigenvector for $\lambda = 1$:

$$eT = eMH^T = eH^T = e. \tag{7.33}$$

The crucial question is the significance of the *right eigenvector* in $Ta = a$. It gives the inner products $\langle \phi(t), \phi(t+k)\rangle$. *Convergence of the cascade algorithm in L^2 becomes convergence of the power method $a^{(i+1)} = Ta^{(i)}$.*

We include this convergence proof in the text because the argument is straightforward. Condition E on T also determines [Cohen-Daubechies] whether the translates $\phi(t+k)$ form a Riesz basis. [Eirola] and [Villemoes] use the same matrix T in a different way, to find the smoothness of $\phi(t)$ (next section).

We work with one-dimensional filters, but the analysis is the same in higher dimensions [Lawton-Lee-Shen]. In a sense our approach completes the convergence analysis of [CDM], by working with $|H(\omega)|^2$ — the autocorrelation of the filter (or mask). The key is to identify T and watch its eigenvalues.

Examples of Divergence and Weak Convergence

Example 7.1. The coefficients $h(k) = 1, \frac{1}{2}, -\frac{1}{2}$ have even sum = odd sum = $\frac{1}{2}$. The dilation equation is

$$\phi(t) = 2\phi(2t) + \phi(2t-1) - \phi(2t-2). \tag{7.34}$$

This looks innocent, but in a few steps the cascade algorithm is a disaster. The value at $t = 0$ is doubled at every iteration. Each step gives $\phi^{(i+1)}(0) = 2\phi^{(i)}(0)$.

This blowup at a point does not by itself rule out finite energy. The function $f(t) = t^{-1/3}$ also blows up at $t = 0$, but on the interval $[0, 1]$ its energy is $\int t^{-2/3}dt = 3$. Therefore this example is continued below, to prove that the energy in $\phi^{(i)}(t)$ does become infinite as $i \to \infty$. When the first coefficient in (7.34) is between $\frac{1}{2}(1-\sqrt{3})$ and $\frac{1}{2}(1+\sqrt{3})$, which allows the possibility of blowup at $t = 0$, the cascade algorithm converges in L^2.

Example 7.2. The coefficients $h(k) = \frac{1}{2}, 0, 0, \frac{1}{2}$ have even sum = odd sum = $\frac{1}{2}$ and also double-shift orthogonality. All the shifted vectors $0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, \ldots$ and $0, 0, 0, 0, \frac{1}{2}, 0, 0, \frac{1}{2}, \ldots$ are orthogonal. The expected solution of the dilation equation is a *stretched box*:

$$\phi(t) = \phi(2t) + \phi(2t-3) \text{ leads to } \phi(t) = \begin{cases} \frac{1}{3} & 0 \le t < 3 \\ 0 & \text{else.} \end{cases}$$

The box area is still one. Two half-boxes still fit into a whole box. But when the cascade algorithm starts with the unit box, it converges only *weakly* to $\phi(t)$.

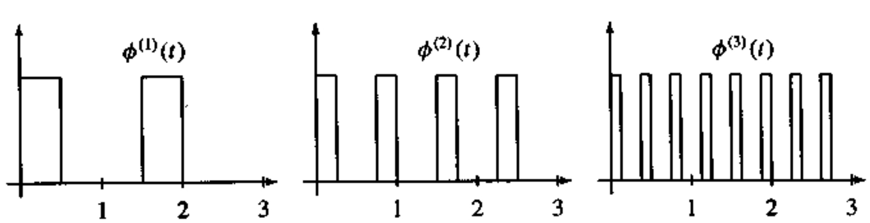

Figure 7.2: Three cascade steps $\phi^{(i+1)}(t) = \phi^{(i)}(2t) + \phi^{(i)}(2t-3)$.

The area between 0 and 3 is one-third filled, more and more densely. At no point in Figure 7.2 do the functions $\phi^{(i)}(t)$ actually equal $\frac{1}{3}$; they always equal 0 or 1. But the area above every interval Δt approaches $\frac{1}{3}\Delta t$. This is weak convergence to $\phi(t) = \frac{1}{3}$, stated for any smooth function $f(t)$:

$$\lim_{i\to\infty}\int_0^3 \phi^{(i)}(t)f(t)\,dt = \int_0^3 \tfrac{1}{3}f(t)\,dt. \tag{7.35}$$

The adjective "weak" means that integrals converge, even if functions themselves do not. Fast oscillations in $\phi^{(i)}(t)$ are averaged out by the integration.

Note that this shifted box $\phi(t)$ is *not orthogonal* to its translates. Furthermore, its energy $\int(\phi(t))^2\,dt$ is $\frac{1}{3}$ instead of 1. Those facts seem surprising, because at every step $\phi^{(i)}(t)$ has unit energy and is orthogonal to all other $\phi^{(i)}(t+k)$. This illustrates the weakness of convergence — inner products of $\phi^{(i)}(t)$ with fixed $f(t)$ converge but inner products with $\phi^{(i)}(t+k)$ do not converge to $\int\phi(t)\phi(t+k)\,dt$.

Our inner product and energy formulas will involve T. This matrix always has $\lambda = 1$ as an eigenvalue, because of the zero at π. Explosive failure as in Example 1 is associated with an eigenvalue that has $|\lambda| > 1$. Weak convergence as in Example 2 is associated with a repeated $\lambda = 1$ and/or other eigenvalues with $|\lambda| = 1$ (with a full set of eigenvectors). Total success occurs when all other eigenvalues of T have $|\lambda| < 1$; this will be Condition E. The smaller those other eigenvalues, the smoother the function $\phi(t)$.

The Inner Product Formula

The energy in a real function $\phi(t)$ (its L^2 norm) is its inner product with itself. We also want its inner product with its translates $\phi(t+k)$. This inner product is $\boldsymbol{a}(k)$:

$$\boldsymbol{a}(k) = \int_{-\infty}^{\infty}\phi(t)\phi(t+k)\,dt, \quad -\infty < k < \infty. \tag{7.36}$$

Note that $\boldsymbol{a}(k) = \boldsymbol{a}(-k)$. The number $\boldsymbol{a}(0)$ is the energy $||\phi||^2$. When $\phi(t)$ is zero outside the interval $0 \le t < N$, the inner products $\boldsymbol{a}(k)$ are all zero for $|k| \ge N$. The translated $\phi(t+N)$ does not overlap $\phi(t)$. Only the $2N-1$ central components of $\boldsymbol{a}$ can be nonzero.

The cascade algorithm starts each step with $\phi^{(i)}(t)$. Suppose we know the inner products $\boldsymbol{a}^{(i)}(k)$ between that function and its translates. Iteration produces a new function $\phi^{(i+1)}(t)$, and we want the new inner products $\boldsymbol{a}^{(i+1)}(k)$. The new function is a combination $\sum 2\boldsymbol{h}(k)\phi^{(i)}(2t-k)$, *with t rescaled to $2t$*. The new inner products depend on the old ones, and on the numbers $\boldsymbol{h}(k)$. We now find the formula that the inner products obey.

Lemma. To find the vector $\boldsymbol{a}^{(i+1)}$ of inner products, multiply $\boldsymbol{a}^{(i)}$ by the Toeplitz matrix $2\boldsymbol{H}\boldsymbol{H}^T$ (which gives $4N-1$ components) and downsample:

$$T\boldsymbol{a}^{(i)} = \boldsymbol{a}^{(i+1)} = (\downarrow 2)2\boldsymbol{H}\boldsymbol{H}^T\boldsymbol{a}^{(i)}. \tag{7.37}$$

Example 7.1 (continued) With $2\boldsymbol{h}(k) = 2, 1, -1$, the numbers along the diagonals of the symmetric matrix $2\boldsymbol{H}\boldsymbol{H}^T$ are $\frac{1}{2}(2,1,-1)*(-1,1,2) = \frac{1}{2}(-2,1,6,1,-2)$. This comes from convolution and also from matrix multiplication: $2\boldsymbol{H}\boldsymbol{H}^T$ is

$$\frac{1}{2}\begin{bmatrix} \cdot & & & & \\ 1 & 2 & & & \\ -1 & 1 & 2 & & \\ & -1 & 1 & 2 & \end{bmatrix}\begin{bmatrix} \cdot & 1 & -1 & & \\ & 2 & 1 & -1 & \\ & & 2 & 1 & \\ & & & 2 & \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \\ -2 & 1 & 6 & 1 & -2 \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

$(\downarrow 2)$ removes the odd-numbered rows to leave double shifts in $\boldsymbol{T}$:

$$\boldsymbol{T} = \frac{1}{2}\begin{bmatrix} \cdot & -2 & & & & & \\ \cdot & 6 & 1 & -2 & & & \\ & -2 & 1 & 6 & 1 & -2 & \\ & & & -2 & 1 & 6 & \cdot \\ & & & & & -2 & \cdot \end{bmatrix}. \tag{7.38}$$

All columns of $\boldsymbol{T}$ *add to* 1! At the first cascade step, the box function $\phi^{(0)}(t)$ has inner products $\boldsymbol{a}^{(0)} = (\dots, 0, 1, 0, \dots)$. The Lemma says that the inner products of $\phi^{(1)}(t)$ with its translates are

$$\boldsymbol{a}^{(1)} = T\boldsymbol{a}^{(0)} = \frac{1}{2}\begin{bmatrix} \cdot \\ 0 \\ -2 \\ 6 \\ -2 \\ 0 \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ 0 \\ -1 \\ 3 \\ -1 \\ 0 \\ \cdot \end{bmatrix}.$$

Since $N = 2$ for this filter $\boldsymbol{H}$, all inner products are zero for $|k| \geq 2$. We only need the center submatrix of order $2N - 1 = 3$, and we iterate again:

$$\boldsymbol{a}^{(2)} = \frac{1}{2}\begin{bmatrix} 1 & -2 & 0 \\ 1 & 6 & 1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -7 \\ 16 \\ -7 \end{bmatrix}. \tag{7.39}$$

Clearly the numbers are growing. The cascade algorithm is diverging. $\boldsymbol{T}$ has an eigenvalue larger than 1. It is $\frac{5}{2}$, and this filter is a disaster in iteration.

Example 7.2 (continued) With coefficients $2\boldsymbol{h}(k) = 1, 0, 0, 1$, the matrix $2\boldsymbol{H}\boldsymbol{H}^T$ has rows containing $\frac{1}{2}(1, 0, 0, 2, 0, 0, 1)$. Downsampling removes every other row to leave double shifts in $\boldsymbol{T}$. Since $N = 3$ and $2N - 1 = 5$, we look only at $\boldsymbol{T}_5$:

$$T_5 = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ 2 & 0 & 0 & 1 & \\ 0 & 0 & 2 & 0 & 0 \\ & 1 & 0 & 0 & 2 \\ & & & 1 & 0 \end{bmatrix}.$$

Again all columns add to 1. Starting again with the box function $\phi^{(0)}(t)$, its inner products are $\boldsymbol{a}^{(0)} = (0, 0, 1, 0, 0)$. *Multiplying by* T_5 *produces this same vector* $\boldsymbol{a}^{(1)} = \boldsymbol{a}^{(0)}$. Therefore $\phi^{(1)}(t)$ is also orthogonal to its translates.

That conclusion is no surprise. The $\boldsymbol{h}(k)$ have double-shift orthogonality. The center column of $\boldsymbol{T}$ agrees with the identity matrix. At every step $\boldsymbol{a}^{(i)} = \boldsymbol{\delta}$.

Still there is weak trouble. The reason is that $\boldsymbol{T}_5$ has a repeated eigenvalue at $\lambda = 1$. Its other eigenvalues are $-1, -\frac{1}{2}, \frac{1}{2}$. A second eigenvector for $\lambda = 1$ is $\frac{1}{9}(1, 2, 3, 2, 1)$. Those are the actual inner products of the stretched ϕ_{box} on the interval $0 \leq t < 3$. The inner products $\boldsymbol{a}^{(i)} = \boldsymbol{\delta}$ do not approach the inner products of ϕ_{box}. The cascade algorithm does not converge in L^2 to this stretched box.

Example 7.3. (Convergence to the hat function) With coefficients $2h(k) = \frac{1}{2}, 1, \frac{1}{2}$, the matrix $2HH^T$ has entries $\frac{1}{8}(1, 4, 6, 4, 1)$. Downsampling leaves $T_{2N-1} = T_3$:

$$T_3 = \frac{1}{8}\begin{bmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{bmatrix} \quad \text{has } \lambda = 1, \tfrac{1}{2}, \tfrac{1}{4}. \tag{7.40}$$

This example is successful but not orthogonal. Each multiplication of $a^{(0)} = (0, 1, 0)$ by T_3 gives the inner products at the next cascade step:

$$a^{(1)} = \frac{1}{8}\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} \quad a^{(2)} = \frac{1}{64}\begin{bmatrix} 10 \\ 44 \\ 10 \end{bmatrix} \quad \cdots \quad a^{(\infty)} = \frac{1}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}. \tag{7.41}$$

We jumped to the limit $a^{(\infty)}$ because it is the eigenvector of T_3 for $\lambda = 1$.

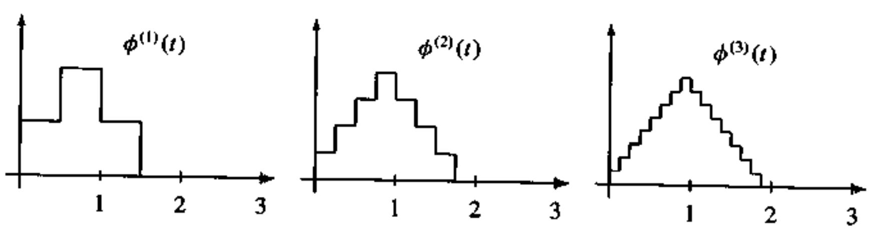

Figure 7.3: Three cascade steps $\phi^{(i+1)}(t) = \frac{1}{2}\phi^{(i)}(2t) + \phi^{(i)}(2t-1) + \frac{1}{2}\phi^{(i)}(2t-2)$.

The limit vector $a^{(\infty)} = a$ contains the inner products of the hat function with its translates. The $\phi^{(i)}(t)$ in Figure 7.3 are converging to $\phi^{(\infty)}(t)$ = hat function, and the inner products $a^{(i)}(k)$ are converging to $a^{(\infty)}(k) = \frac{1}{6}, \frac{4}{6}, \frac{1}{6}$. We are seeing the **power method** in operation, for the functions and also for the vectors. The hat function is the steady-state fixed point of the operator in the cascade. The vector a is the steady-state eigenvector of T with $\lambda = 1$.

Proof of the Lemma To show that the inner products are $a^{(i+1)} = Ta^{(i)}$, it is very convenient to compute all $a^{(i+1)}(k)$ at once. *Thus we work with vectors*:

$$\Phi^{(i)}(t) = \begin{bmatrix} \cdot \\ \phi^{(i)}(t-1) \\ \phi^{(i)}(t) \\ \phi^{(i)}(t+1) \\ \cdot \end{bmatrix} \quad \text{and} \quad \Phi(t) = \begin{bmatrix} \cdot \\ \phi(t-1) \\ \phi(t) \\ \phi(t+1) \\ \cdot \end{bmatrix}. \tag{7.42}$$

The next vector in the cascade is $\Phi^{(i+1)}(t) = (\downarrow 2)2H\Phi^{(i)}(2t)$. The rescaling to $2t$ is accounted for by $(\downarrow 2)$. This is the way to a vector-based calculation:

$$\begin{aligned} a^{(i+1)} &= \int_{-\infty}^{\infty} \phi^{(i+1)}(t)\ \Phi^{(i+1)}(t)\,dt \\ &= \int_{-\infty}^{\infty} \left[2\sum h(k)\phi^{(i)}(2t-k)\right]\left[(\downarrow 2)2H\Phi^{(i)}(2t)\right]dt. \end{aligned} \tag{7.43}$$

Bring the operator $(\downarrow 2)2H$ outside the integral. Change variables in the kth term to $u = 2t - k$. That kth term of the integral becomes (with $du = 2\,dt$)

$$\int_{-\infty}^{\infty} h(k)\phi^{(i)}(u)\Phi^{(i)}(u+k)du = h(k)S^{-k}a^{(i)}. \tag{7.44}$$

The k-step shift S^{-k} allowed us to write $\Phi^{(i)}(u+k)$ as $S^{-k}\Phi^{(i)}(u)$. Then the integration with respect to u produced $a^{(i)}$. Now sum equation (7.44) on k to reach the matrix $\sum h(k)S^{-k}$, which is H^T as the Lemma requires:

$$a^{(i+1)} = (\downarrow 2)2HH^T a^{(i)} = Ta^{(i)}. \tag{7.45}$$

Corollary *The inner products $a(k)$ of the scaling function $\phi(t)$ with $\phi(t+k)$ are in the eigenvector of T corresponding to $\lambda = 1$:*

$$a = Ta. \tag{7.46}$$

This assumes that the scaling function exists in L^2, which we will prove. To reach $a = Ta$, repeat the calculation above without the superscripts.

The power method $a^{(i)} = T^i a^{(0)}$ converges when T has a non-repeated eigenvalue $\lambda = 1$ and all other eigenvalues have $|\lambda| < 1$. This "Condition E" gives $a^{(i)} \to a$. (Note! All vectors are normalized by $\sum a(k) = 1$.) Convergence for the functions $\phi^{(i)}(t)$ is still to prove, but convergence for their inner products $a^{(i)}$ is easier — just linear algebra.

Theorem 7.7 *The infinite matrix $T = (\downarrow 2)2HH^T$ and its submatrix T_{2N-1} always have $\lambda = 1$ as an eigenvalue. The power iteration $a^{(i+1)} = Ta^{(i)}$ converges to the eigenvector $a = Ta$ if and only if T_{2N-1} satisfies*

Condition E: *T_{2N-1} has all $|\lambda| < 1$ except for a simple eigenvalue $\lambda = 1$.*

Proof. Suppose the starting $a^{(0)}$ is expanded as a combination $a + c_2v_2 + c_3v_3 + \cdots$ of eigenvectors of T_{2N-1}. Every time we multiply by that matrix, each v_j is multiplied by the corresponding λ_j. Since $|\lambda_j| < 1$ by Condition E, those components get smaller. In the limit as $i \to \infty$, the powers $T^i a^{(0)}$ converge to the eigenvector a — whose coefficient stays at 1 (because it has $\lambda = 1$).

This proof only works if T_{2N-1} has a full set of eigenvectors, to expand $a^{(0)}$. To cover all cases we use the Jordan form of T_{2N-1}. It has $\lambda = 1$ alone in a 1×1 block. All other blocks have $|\lambda| < 1$ and their powers approach zero.

Convergence of the Cascade Algorithm in L^2

The convergence proof will be easy if we know that the dilation equation $\phi(t) = \sum 2h(k)\phi(2t - k)$ has a finite energy solution. We now prove that $\phi^{(i)}(t)$ converges to this scaling function $\phi(t)$. Properly speaking, we must also show the existence of $\phi(t)$ itself. This existence step is logically first but it will come later for simplicity.

Theorem 7.8 *Assume that $\phi(t)$ is in L^2. The cascade sequence $\phi^{(i)}(t)$ converges to $\phi(t)$ if and only if T satisfies Condition E. Then*

$$\|\phi^{(i)} - \phi\|^2 = \|\phi^{(i)}\|^2 - 2 < \phi^{(i)}, \phi > + \|\phi\|^2 = a^{(i)}(0) - 2b^{(i)}(0) + a(0) \tag{7.47}$$

$$\textit{converges to} \quad a(0) - 2a(0) + a(0) = 0.$$

Proof. The numbers $a^{(i)}(0)$ and $a(0)$ are the energies $||\phi^{(i)}(t)||^2$ and $||\phi(t)||^2$:

$$\int \phi^{(i)}(t)\phi^{(i)}(t+0)\,dt = a^{(i)}(0) \quad \text{and} \quad \int \phi(t)\phi(t+0)\,dt = a(0).$$

We know already that $a^{(i)}$ converges to a. This was the preceding theorem. Equation (7.47) also contains the inner product of $\phi^{(i)}(t)$ with $\phi(t)$. This is the zeroth component $b^{(i)}(0)$ of a new vector of inner products $b^{(i)}(k) = \int \phi^{(i)}(t)\phi(t+k)\,dt$.

Our main calculation found each vector $a^{(i+1)}$ from the previous $a^{(i)}$. The rule was to multiply by T. Condition E gave convergence to a. In the same way, we now show that $b^{(i+1)} = Tb^{(i)}$. Then the vectors $b^{(i)}$ converge (this is the power method again) to the same eigenvector a. Therefore $-2b^{(i)}(0)$ in (7.47) converges to $-2a(0)$, which completes the proof that $||\phi^{(i)} - \phi||^2$ converges to zero.

For the new calculation $b^{(i+1)} = Tb^{(i)}$, it is again convenient to work with vectors:

$$b^{(i+1)} = \int_{-\infty}^{\infty} \phi^{(i+1)}(t) \begin{bmatrix} \cdot \\ \phi(t-1) \\ \phi(t) \\ \phi(t+1) \\ \cdot \end{bmatrix} dt = \int_{-\infty}^{\infty} \phi^{(i+1)}(t)\Phi(t)\,dt.$$

Substitute the cascade formula for $\phi^{(i+1)}(t)$ and the dilation equation for $\Phi(t)$:

$$b^{(i+1)} = \int_{-\infty}^{\infty} \Big[2\sum_k h(k)\phi^{(i)}(2t-k)\Big]\,[(\downarrow 2)2H\Phi(2t)]\,dt. \tag{7.48}$$

This matches equation (7.43) when $\Phi^{(i)}$ is replaced by Φ. Change variables in the kth term to $u = 2t - k$, and that term matches (7.44)—with a replaced by b. Then sum on k to match equation (7.45). These same steps give the new answer, with a changed to b:

$$b^{(i+1)} = (\downarrow 2)2HH^T b^{(i)} = Tb^{(i)}. \tag{7.49}$$

The normalization $\sum b^{(0)}(k) = 1$ is still true, because $\int \phi^{(i)}(t)\sum \phi(t+k)\,dt = \int \phi^{(i)}\,dt = 1$. So the power method starting from $b^{(0)}$ converges to the same vector a (not b!). By equation (7.47) the sequence $\phi^{(i)}(t)$ converges to $\phi(t)$.

The converse is also straightforward [Strang3]. Convergence of the functions $\phi^{(i)}$ to ϕ implies convergence of their inner products $a^{(i)}$ to a. Thus the power method always goes to the same limiting vector a. Condition E must hold.

Existence of the Scaling Function

This is not a book about proofs. But the dilation equation is so fundamental that we must be sure it has a solution. Formally the infinite product $\prod H(\omega/2^j)$ yields the Fourier transform $\widehat{\phi}(\omega)$. The real question is when this product converges strongly to a finite energy solution, and the answer to that question is Condition E.

We base existence of $\phi(t)$ on our calculation of inner products, which is the key to this section. The space L^2 is *complete*, so when we prove that the energies $\|\phi^{(m)} - \phi^{(n)}\|^2$ approach zero, there is guaranteed to exist a limit function $\phi(t)$ in L^2.

Theorem 7.9 *If Condition E holds then $\|\phi^{(m)} - \phi^{(n)}\|^2$ approaches zero as $m, n \to \infty$. Therefore the sequence $\phi^{(0)}(t), \phi^{(1)}(t), \ldots$ converges to a limit $\phi(t)$ in L^2.*

Proof. The energy $\|\phi^{(m)} - \phi^{(n)}\|^2$ is $\|\phi^{(m)}\|^2 - 2\langle\phi^{(m)}, \phi^{(n)}\rangle + \|\phi^{(n)}\|^2$. The first and third terms are $\boldsymbol{a}^{(m)}(0)$ and $\boldsymbol{a}^{(n)}(0)$, both approaching the limit $\boldsymbol{a}(0)$. We must show that the inner product $\langle\phi^{(m)}, \phi^{(n)}\rangle$ also approaches $\boldsymbol{a}(0)$.

Suppose $m = i + n$ with fixed $i > 0$. The same inner product calculation, following the pattern (7.43)–(7.44)–(7.45) and multiplying by $\boldsymbol{T}$ at each step, yields

$$\langle\phi^{(m)}, \phi^{(n)}\rangle = \boldsymbol{T}^n\langle\phi^{(i)}, \phi^{(0)}\rangle = \boldsymbol{T}^n\boldsymbol{c}^{(i)}(0). \tag{7.50}$$

The vector $\boldsymbol{c}^{(i)}$ contains the inner products $\boldsymbol{c}^{(i)}(k) = \int \phi^{(i)}(t)\phi^{(0)}(t+k)\,dt$. For each fixed i, the power method $\boldsymbol{T}^n\boldsymbol{c}^{(i)}$ approaches $\boldsymbol{a}$. The limit of (7.50) is $\boldsymbol{a}(0)$ as desired. But the difficulty (since $i = m - n$ is arbitrary) is that this convergence must hold *uniformly for all starting vectors* $\boldsymbol{c}^{(i)}$. We know two facts about the components $\boldsymbol{c}^{(i)}(k)$. They are uniformly bounded and they add to 1:

$$|\boldsymbol{c}^{(i)}(k)| \le \|\phi^{(i)}\|\,\|\phi^{(0)}\| \le \|\boldsymbol{a}^{(i)}\| \le C \tag{7.51}$$

$$\sum_k \boldsymbol{c}^{(i)}(k) = \int_{-\infty}^{\infty} \phi^{(i)}(t)\sum \phi^{(0)}(t+k)\,dt = \int_{-\infty}^{\infty} \phi^{(i)}(t)\,dt = 1. \tag{7.52}$$

Condition E and the Jordan form $\boldsymbol{J}$ of $\boldsymbol{T}$ give uniform convergence $\boldsymbol{T}^n\boldsymbol{c} \to \boldsymbol{a}$:

$$\boldsymbol{T}^n\boldsymbol{c} = \boldsymbol{S}\boldsymbol{J}^n\boldsymbol{S}^{-1}\boldsymbol{c} = \begin{bmatrix} \boldsymbol{a} & -- \end{bmatrix}\begin{bmatrix} 1 & \\ & \boldsymbol{B}^n \end{bmatrix}\begin{bmatrix} \boldsymbol{e} \\ --- \end{bmatrix}\begin{bmatrix} \boldsymbol{c} \end{bmatrix}. \tag{7.53}$$

The left eigenvector $\boldsymbol{e} = [1\ 1\ \ldots\ 1]$ is in row 1, and $\boldsymbol{a}$ in column 1 is the right eigenvector. These eigenvector matrices $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$ are fixed, and the block $\boldsymbol{B}$ has eigenvalues $|\lambda| < 1$. Therefore $\boldsymbol{B}^n \to \boldsymbol{0}$ and we have uniform convergence to $\boldsymbol{aec} = \boldsymbol{a}\sum \boldsymbol{c}(k) = \boldsymbol{a}$. Equation (7.50) approaches $\boldsymbol{a}(0)$ as m and n get large, completing the proof that $||\phi^{(m)} - \phi^{(n)}||^2 \to 0$.

Remark 1 The convergence of the cascade algorithm can be interpreted in the frequency domain, which is illuminating. The convergence of $\phi^{(i)}(t)$ to $\phi(t)$ becomes convergence of the infinite product to $\widehat{\phi}(\omega)$:

$$\widehat{\phi}^{(i)}(\omega) = \Big[\prod_{j=1}^{i} H(\omega/2^j)\Big]\widehat{\phi}^{(0)}(\omega/2^i) \ \text{ converges in } L^2 \text{ to } \ \widehat{\phi}(\omega) = \prod_{j=1}^{\infty} H(\omega/2^j).$$

Remark 2 For filters with double-shift orthogonality, there is no danger that $\boldsymbol{T}$ has an eigenvalue with $|\lambda| > 1$. The norm of $\boldsymbol{T}$ is $sup(|H(\omega)|^2 + |H(\omega+\pi)|^2)$ and this is 1. Condition E reduces to the Cohen-Lawton condition that $\lambda = 1$ is a simple eigenvalue of $\boldsymbol{T}$. Then $\{\phi(t+k)\}$ is an *orthonormal* basis and $\boldsymbol{a} = \boldsymbol{\delta}$.

Condition E holds in this orthogonal case if $H(\omega) \neq 0$ for $|\omega| \le \frac{\pi}{3}$ [Mallat1, JiaWang].

Remark 3 The same method (7.43)-(7.44)-(7.45) that gives inner products of scaling functions also gives inner products with wavelets. The highpass operator $\boldsymbol{H}_1$ replaces the lowpass $\boldsymbol{H}$ in the appropriate places:

$$\langle\phi(t), w(t+k)\rangle = (\downarrow 2)2\boldsymbol{H}\boldsymbol{H}_1^T\boldsymbol{a} \tag{7.54}$$

$$\langle w(t), w(t+k)\rangle = (\downarrow 2)2\boldsymbol{H}_1\boldsymbol{H}_1^T\boldsymbol{a} \tag{7.55}$$

These are derived in Section 11.6. *We find inner product formulas for any functions that satisfy two-scale equations.*

Remark 4 For "multifilters" the coefficients $\boldsymbol{h}(k)$ are $r \times r$ matrices. The dilation equation determines a *vector* of r scaling functions. The inner product $\int \boldsymbol{\phi}(t)\boldsymbol{\phi}^T(t+k)\,dt$ is an $r \times r$ matrix $\boldsymbol{a}(k)$. The equation $\boldsymbol{Ta} = \boldsymbol{a}$ for a vector of matrices now involves a matrix convolution:

$$\boldsymbol{Ta} = (\downarrow 2)2\boldsymbol{h} * \boldsymbol{a} * \boldsymbol{h}^T. \tag{7.56}$$

In the frequency domain this *matrix transition operator* $\boldsymbol{T}$ becomes

$$\boldsymbol{TA}(\omega) = \boldsymbol{H}\left(\frac{\omega}{2}\right)\boldsymbol{A}\left(\frac{\omega}{2}\right)\boldsymbol{H}^*\left(\frac{\omega}{2}\right) + \boldsymbol{H}\left(\frac{\omega}{2}+\pi\right)\boldsymbol{A}\left(\frac{\omega}{2}+\pi\right)\boldsymbol{H}^*\left(\frac{\omega}{2}+\pi\right). \tag{7.57}$$

The theory develops on the same lines [Cohen-Daubechies-Plonka] to give the existence in L^2, the smoothness, the approximation properties, and the stability of the basis $\{\phi_i(t+k)\}$. The eigenvalues of $\boldsymbol{T}$ are still in control.

Problem Set 7.2

1. If $\boldsymbol{h}(k)$ has double-shift orthogonality (Condition O), show that the central column of $\boldsymbol{T}$ is $\boldsymbol{\delta}$. This is an eigenvector of $\boldsymbol{T}$ for $\lambda = 1$.
2. Construct the finite matrices $\boldsymbol{T}$ for $\boldsymbol{h} = \frac{1}{3}(1, 1, 1)$ and $\boldsymbol{h} = (c, \frac{1}{2}, \frac{1}{2}, -c)$.
3. Draw the output from one Haar cascade step $\phi^{(1)}(t) = \phi^{(0)}(2t) + \phi^{(0)}(2t-1)$ with
   (a) $\phi^{(0)}(t)$ = unit box on the interval $[1, 2]$
   (b) $\phi^{(0)}(t)$ = hat function on the interval $[0, 2]$
   (c) $\phi^{(0)}(t)$ = hat function on the interval $[0, 1]$.

   Two of those $\phi^{(0)}(t)$ lead to convergence. Which one doesn't?
4. Prove that $P^{(1)}(t) = \sum \phi^{(1)}(t-n)$ equals $P^{(0)}(2t) = \sum \phi^{(0)}(2t-n)$ for any starting function $\phi^{(0)}(t)$ and any coefficients $\boldsymbol{h}(k)$ with a zero at π: even sum = odd sum = $\frac{1}{2}$.

   It follows that $P^{(i)}(t) = P^{(0)}(2^i t)$ will oscillate faster and faster. *There is no convergence unless* $P^{(0)}(t) = \sum \phi^{(0)}(t-n)$ *is identically one.* This is the condition on $\phi^{(0)}(t)$ in the cascade algorithm.
5. Find the eigenvalues of $\boldsymbol{T}$ in Problem 2 (depending on c).
6. Show that the filter $\boldsymbol{h} = [-1\ \ 3\ \ 3\ \ -1]/4$ does *not* satisfy Condition E, and $\boldsymbol{T}$ has $\lambda = 2.1712$. This bad $\boldsymbol{h}$ is dual to the good spline filter $\boldsymbol{f} = [1\ \ 3\ \ 3\ \ 1]/8$; their product $\boldsymbol{h} * \boldsymbol{f}$ is Daubechies maxflat halfband.

7.3 Smoothness of Scaling Functions and Wavelets

The previous section established Condition E for the convergence of the cascade algorithm. The eigenvalues of $\boldsymbol{T}$ are required to be less than 1 (except the simple eigenvalue $\lambda = 1$). The limit function $\phi(t)$ is then in L^2—which assures some minimal smoothness, but not much. If the eigenvalues of T are less than 4^{-s}, apart from the special eigenvalues that are powers of $\frac{1}{2}$, *we now show that $\phi(t)$ and $w(t)$ have s derivatives.*

This conclusion is true whether s is an integer or not. The proof for integers s is very direct, so we include it. The proof for noninteger s needs more space and effort, so we refer for example to [Villemoes]. These are derivatives in the L^2 sense because we work with the matrix T. In the frequency domain, each derivative is a multiplication by $i\omega$. Therefore $\phi(t)$ in L^2 has s derivatives in L^2 when

$$||\phi^{(s)}(t)||^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^{2s}|\widehat{\phi}(\omega)|^2 d\omega \text{ is finite.}$$

This definition allows s to be a fraction (or negative) with no difficulty. Since $w(t)$ is a combination of $\phi(2t-k)$, we only need to study the smoothness of $\phi(t)$.

The basic idea is simple. *Each new factor $\left(\frac{1+z^{-1}}{2}\right)$ in $H(z)$ has four effects*:

1. All eigenvalues of T are divided by 4.
2. The old $\phi(t)$ is convolved with the box function.
3. The old $\widehat{\phi}(\omega)$ is multiplied by $\left(1-e^{-i\omega}\right)/i\omega$.
4. The new $\phi(t)$ has *one more derivative* than the old $\phi(t)$.

When we check these facts, our desired result is proved. You might think that the final $\phi(t)$ has the full p derivatives, because $H(z)$ has p factors of $\frac{1+z^{-1}}{2}$. But some of those factors are needed to get the eigenvalues of T below 1 (always excluding the special eigenvalues $1, \frac{1}{2}, \ldots$). If s (integer) is less than the number $s_{\max}$ below, there will be s factors still left after this Condition **E** is met. Then $\phi(t)$ has s derivatives. The non-special eigenvalues are below 4^{-s}.

The word "regularity" has been applied to s and also to p. *Those are different numbers* (we will prove $s < p$). So we avoid that word, and refer to smoothness s and accuracy p. We state the conclusions for any s; our proof was for s = integer.

Theorem 7.10 *Each new factor $\frac{1+z^{-1}}{2}$ has the effects* **1, 2, 3, 4**. *Then $\phi(t)$ and $w(t)$ have s derivatives in L^2 when the non-special eigenvalues of T have $|\lambda| < 4^{-s}$. The supremum $s_{\max}$, when $\phi(t)$ comes from $H(z) = \left(\frac{1+z^{-1}}{2}\right)^p Q(z)$, is*

$$s_{\max} = p - \log_4 \left|\lambda_{\max}\left(T_Q\right)\right| \quad \textit{with} \quad T_Q = (\downarrow 2)2QQ^T. \tag{7.58}$$

A short MATLAB code will construct T or T_Q from h and find its eigenvalues. In practice, we exclude the special $\lambda = 1, \ldots, (\frac{1}{2})^{2p-1}$ directly from the eigenvalues of T, and then $s_{\max} = -\log_4(|\lambda_{\max}(T)|)$.

Actually **1, 2, 3** are already proved. The effect on the eigenvalues of T was established in Section 7.2. Each eigenvalue is divided by 2 twice, because T comes from $H(z)H(z^{-1})$. So all eigenvalues of T_{old} are divided by 4. Then T_{new} also has the special eigenvalues 1 and $\frac{1}{2}$. The extra $\frac{1+z^{-1}}{2}$ in $H(z)$ changes the infinite product to $\widehat{\phi}_{\text{new}}(\omega)$:

$$\widehat{\phi}_{\text{new}}(\omega) = \prod_{j=1}^{\infty}\left(\tfrac{1}{2}+\tfrac{1}{2}e^{-i\omega/2^j}\right)\widehat{\phi}_{\text{old}}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)\widehat{\phi}_{\text{old}}(\omega). \tag{7.59}$$

The extra "Haar factors" multiply to give the "box factor." This is the infinite product in Section 6.4 that yields the box function $B(t)$. In the time domain $\phi_{\text{new}}(t)$ is the box convolved with $\phi_{\text{old}}(t)$. This adds one more derivative for ϕ_{new}.

Lemma 7.1 *The convolution $\phi_{\text{new}}(t) = B(t) * \phi_{\text{old}}(t)$ has $s+1$ derivatives if and only if $\phi_{\text{old}}(t)$ has s derivatives.*

Proof. In frequency we are multiplying $\widehat{\phi}_{\text{old}}(\omega)$ by $\left(1-e^{-i\omega}\right)/i\omega$. This has magnitude at most $2/|\omega|$. Therefore $\widehat{\phi}_{\text{new}}$ decreases at least one order faster than $\widehat{\phi}_{\text{old}}$:

$$\int_{-\infty}^{\infty} |\omega|^{2(s+1)} \left|\widehat{\phi}_{\text{new}}(\omega)\right|^2 d\omega \leq 4 \int_{-\infty}^{\infty} |\omega|^{2s} \left|\widehat{\phi}_{\text{old}}(\omega)\right|^2 d\omega.$$

The factor 4 comes from $(2/|\omega|)^2$ times $|\omega|^2$. The last integral is finite when ϕ_{old} has s derivatives. So the first integral is finite and ϕ_{new} has $s+1$ derivatives. In the time domain, the derivative of the convolution $\phi_{\text{new}}(t) = B(t) * \phi_{\text{old}}(t)$ is a *difference*:

$$\phi'_{\text{new}}(t) = \frac{d}{dt}\int_0^1 \phi_{\text{old}}(t-s)\,ds = \int_0^1 \phi'_{\text{old}}(t-s)\,ds = \phi_{\text{old}}(t) - \phi_{\text{old}}(t-1). \qquad (7.60)$$

Again ϕ_{new} has one more derivative than ϕ_{old}. This is true in the pointwise sense as well as the L^2 sense. The smoothness increases by one from the extra zero at π in the filter.

For completeness we prove the converse, following [Villemoes]. If ϕ_{new} has $s+1$ derivatives then certainly ϕ'_{new} has s derivatives. From (7.60) this means that $\phi_{\text{old}}(t) - \phi_{\text{old}}(t-1)$ has s derivatives. Use this fact N times:

$$\phi_{\text{old}}(t) - \phi_{\text{old}}(t-N) = [\phi_{\text{old}}(t) - \phi_{\text{old}}(t-1)] + [\phi_{\text{old}}(t-1) - \phi_{\text{old}}(t-2)] + \cdots$$

Each difference on the right has s derivatives. So does the difference on the left. But $\phi(t)$ does not overlap $\phi(t-N)$, so $\phi_{\text{old}}(t)$ by itself must have s derivatives.

Example 7.4. For any $s < \frac{1}{2}$, the box function has s derivatives in L^2.

Reason: $\left|(1-e^{-i\omega})/i\omega\right|$ is below and often near $\frac{2}{|\omega|}$. The integral $\int |\omega|^{2s}\left|\frac{2}{\omega}\right|^2 d\omega$ is finite for $s < \frac{1}{2}$ but infinite for $s = \frac{1}{2}$. Therefore the value $s_{\max} = \frac{1}{2}$ is not actually achieved, which is typical. Normally $\phi(t)$ has s derivatives for all $s < s_{\max}$.

We can check formula (7.58). The box filter $H(z) = \frac{1+z^{-1}}{2}$ has $Q(z) \equiv 1$. Then $Q = I$ and $T_Q = (\downarrow 2)2I$ has $\lambda_{\max} = 2$. The logarithm of 2 to base 4 is $\frac{1}{2}$. Formula (7.58) correctly gives $s_{\max} = 1 - \frac{1}{2} = \frac{1}{2}$.

The splines of degree $p-1$, which come from $p-1$ additional convolutions of the box, have $s_{\max} = p - \frac{1}{2}$. *This is the largest possible* $s_{\max}$! With p zeros at π, $\phi(t)$ cannot have more than $p - \frac{1}{2}$ derivatives in L^2. We will stay with integers to prove that $s \leq p-1$, after noting how the smoothness of splines drops by $\frac{1}{2}$ when we change from L^2 to pointwise.

Pointwise, the box function has zero smoothness. The hat function has one derivative (only one-sided, because the slope jumps). The spline of degree $p-1$ has $p-1$ one-sided derivatives. The general theory says that pointwise smoothness for every function is between $s_{\max} - \frac{1}{2}$ and $s_{\max}$ (this is a *Sobolev inequality*). For splines the pointwise smoothness is at the low end of that range, which is $p-1$.

Theorem 7.11 *If $\phi(t)$ has s derivatives in L^2* (integer s) *then $s < p$. Allowing fractions, $s_{\max}$ cannot exceed $p - \frac{1}{2}$. Pointwise, the smoothness cannot exceed $p-1$.*

Proof. For $\phi(t)$ in L^2 we need at least one zero at π ($H(\pi) = 0$ by Chapter 6). For $\phi'(t)$ in L^2 we need at least two zeros at π (use the Lemma). Continuing, s derivatives in L^2 (*integer s*) require at least $s + 1$ zeros at π. Therefore $p \geq s + 1$ and $s < p$.

These conclusions are consistent with the sth derivative of the dilation equation:

$$\phi^{(s)}(t) = 2^s \sum 2h(k)\phi^{(s)}(2t - k). \tag{7.61}$$

The values $\phi^{(s)}(n)$ at the integers come from the eigenvalue problem $\Phi^{(s)} = 2^s M\Phi^{(s)}$. We know that M has eigenvalue 2^{-s} provided $s < p$. For higher derivatives we cannot even find values at the integers.

We caution that $s < p$ only means that equation (7.61) *might* produce an sth derivative in L^2. It might not! Splines have the highest smoothness that p allows. Compare splines with other scaling functions when $p = 2$. The hat function has $s_{\max} = \frac{3}{2}$, the Daubechies function only has $s_{\max} = 1$. It has .55 derivatives in the pointwise sense [Daubechies-Lagarias]. Smoother wavelets have been constructed (orthogonal or biorthogonal). The neat fact is that the eigenvalues of T immediately give $s_{\max}$.

The Daubechies coefficients are $1+\sqrt{3}, 3+\sqrt{3}, 3-\sqrt{3}, 1-\sqrt{3}$ (divided by 8). The product $H(z)H(z^{-1})$ has coefficients $-1, 0, 9, 16, 9, 0, -1$ (divided by 16). This halfband property gives $(\ldots, 0, 1, 0, \ldots)$ in the center column of T:

$$T_5 = \frac{1}{16}\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 16 & 9 & 0 & -1 & 0 \\ 0 & 9 & 16 & 9 & 0 \\ 0 & -1 & 0 & 9 & 16 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{has eigenvalues} \quad 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}.$$

The eigenvector $a = (0, 0, 1, 0, 0)$ says that all $\phi(t - n)$ are orthogonal to $\phi(t)$. The approximation order is $p = 2$, since $H(\omega)$ has a double zero at π and $(\frac{1}{2})^4$ is not an eigenvalue of T. *The smoothness index is* $s_{\max} = 2 - 1 = 1$ *because the largest "other eigenvalue" is the repeated* $\lambda = 4^{-1}$. The function $\phi(t)$ has *almost* one derivative with finite energy. The integral of $|\omega|^2|\widehat{\phi}(\omega)|^2$ is not finite, but any smaller power of $|\omega|$ will make it finite.

The smoothness of D_6, with a triple zero at π, turns out to be $s_{\max} = 3 - \frac{\log 3}{\log 2}$. With accuracy p, the number of Daubechies coefficients is $2p$. Eirola computed the smoothness of all Daubechies functions up to D_{40} which has $p = 20$, and also found the asymptotic formula:

p	$s_{\max}$	p	$s_{\max}$	Asymptotically
1	0.5	6	2.388	$s_{\max} \approx 0.2075\ p + \text{constant}$
2	1.0	7	2.658	
3	1.415	8	2.914	
4	1.775	9	3.161	
5	2.096	10	3.402	

Cascade Algorithm in the Frequency Domain

When t is rescaled to $2t$, the Fourier transform rescales to $\frac{\omega}{2}$. The shift by k that usually produces $e^{-ik\omega}$ now produces $e^{-ik\omega/2}$. The cascade equation $\phi^{(i+1)}(t) = \sum 2h(k)\phi^{(i)}(2t - k)$ transforms into

$$\widehat{\phi}^{(i+1)}(\omega) = \left(\sum h(k)e^{-ik\omega/2}\right)\widehat{\phi}^{(i)}\left(\frac{\omega}{2}\right) = H\left(\frac{\omega}{2}\right)\widehat{\phi}^{(i)}\left(\frac{\omega}{2}\right). \tag{7.62}$$

The first cascade step multiplies by $H\left(\frac{\omega}{2}\right)$ and the next step by $H\left(\frac{\omega}{4}\right)$. Thus $\widehat{\phi}^{(2)}(\omega)$ is $H\left(\frac{\omega}{4}\right)H\left(\frac{\omega}{2}\right)\widehat{\phi}^{(0)}\left(\frac{\omega}{4}\right)$. The output at step i involves $H^{(i)}$ with i factors:

$$\widehat{\phi}^{(i)}(\omega) = \left[\prod_{j=1}^{i} H\left(\frac{\omega}{2^j}\right)\right]\widehat{\phi}^{(0)}\left(\frac{\omega}{2^i}\right) = H^{(i)}(\omega)\,\widehat{\phi}^{(0)}\left(\frac{\omega}{2^i}\right). \tag{7.63}$$

We expect that this i-term product approaches the infinite product

$$\widehat{\phi}(\omega) = \text{ limit of } \widehat{\phi}^{(i)}(\omega) = \prod_{j=1}^{\infty} H\left(\frac{\omega}{2^j}\right). \tag{7.64}$$

The question is: *When does this limit exist and how smooth is $\phi(t)$?* At each separate frequency ω, the limit exists. That requires only $H(0) = 1$ and a bound C on the derivative $|H'(\omega)|$. Then

$$\left|H\left(\frac{\omega}{2^j}\right)\right| \leq 1 + C\frac{|\omega|}{2^j} \leq e^{C|\omega|/2^j}.$$

If we take logarithms, to look at a sum instead of a product, that sum converges like $\sum C|\omega|/2^j$. The sum $\log|\widehat{\phi}(\omega)|$ is less than $C|\omega|$. Therefore the product $|\widehat{\phi}(\omega)|$ is less than $e^{C|\omega|}$.

Such a bound is useless for large ω! For $\phi(t)$ to be a reasonable function, we need $\widehat{\phi}(\omega)$ to decay rather than grow as $|\omega| \to \infty$. Working with the energy $\int |\widehat{\phi}(\omega)|^2 d\omega$, there is no doubt that each iteration $\widehat{\phi}(\omega)$ retains finite energy. From step i to $i+1$ the energy grows by no more than

$$\|\widehat{\phi}^{(i+1)}\| \leq \|T\|\,\|\widehat{\phi}^{(i)}\| \quad \text{where} \quad \|T\|^2 = \max\left(|H(\omega)|^2 + |H(\omega+\pi)|^2\right). \tag{7.65}$$

The orthonormal case has $\|T\| = 1$, by Condition O. The energy is the same at every iteration of the cascade algorithm. For the biorthogonal case we expect $\|T\| > 1$ and the bound in (7.65) also becomes useless. To find the new energy $\|\widehat{\phi}^{(i+1)}\|^2$ from $\|\widehat{\phi}^{(i)}\|^2$, we look again at the operator T — which is absolutely fundamental to the theory of wavelets.

In the time domain, T is a double-shift Toeplitz matrix. That double shift corresponds to picking out *even frequencies* $0, 2\omega, 4\omega$. In the z-domain it corresponds to picking out *even powers* of z:

Definition 7.1 The transition operator T in the three domains is:

$$\begin{aligned}
Ta(k) &= (\downarrow 2)2\boldsymbol{H}\boldsymbol{H}^T a(k) \\
TA(2\omega) &= |H(\omega)|^2A(\omega) + |H(\omega+\pi)|^2A(\omega+\pi) \\
&= \text{even frequencies in } 2|H(\omega)|^2A(\omega) \\
TA(z^2) &= H(z)A(z)H(z^{-1}) + H(-z)A(-z)H(-z^{-1}) \\
&= \text{even powers in } 2H(z)A(z)H(z^{-1}).
\end{aligned}$$

$A(\omega)$ is the function $\sum a(k)e^{-ik\omega}$ and $A(z)$ is the function $\sum a(k)z^{-k}$. In our application, the $a(k)$ are inner products and $A(\omega)$ is taken from Section 6.4:

$$a(k) = \int \phi(t)\phi(t+k)\,dt \quad \text{and} \quad A(\omega) = \sum \left|\widehat{\phi}(\omega + 2\pi\ell)\right|^2.$$

We give an example immediately, and ask for more in the problem set.

Example 7.5. The coefficients $\boldsymbol{h}(k) = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ give

$$H(\omega) = \tfrac{1}{4}(1 + 2e^{-i\omega} + e^{-2i\omega}) \quad \text{and} \quad H(z) = (1 + 2z^{-1} + z^{-2}).$$

Suppose $\phi^{(0)}(t)$ is the box function, so its inner products are $\mathbf{a}^{(0)}(k) = (\ldots, 0, 1, 0, \ldots)$. The corresponding $A^{(0)}(\omega)$ is the constant function 1. The first step of the cascade algorithm produces $\phi^{(1)}(t)$ as *three half-boxes* with heights $\frac{1}{2}, 1, \frac{1}{2}$. The new energy $\|\phi^{(1)}\|^2$ is $\frac{1}{2}(\frac{1}{4} + 1 + \frac{1}{4}) = \frac{6}{8}$. This should agree with $\boldsymbol{a}^{(1)}(0)$ after the action of $\boldsymbol{T}$:

$$\text{Time:} \qquad \boldsymbol{T}\,\boldsymbol{a}^{(0)} = \frac{1}{8}\begin{bmatrix} 4 & 1 & 0 \\ 4 & 6 & 4 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\text{Frequency:} \quad A^{(1)}(2\omega) &= \text{even frequencies in } 2|H(\omega)|^2 A^{(0)}(\omega) \\
&= \text{even frequencies in } \tfrac{1}{8}\left(e^{2i\omega} + 4e^{i\omega} + 6 + 4e^{-i\omega} + e^{-2i\omega}\right) \\
A^{(1)}(\omega) &= \tfrac{1}{8}\left(e^{i\omega} + 6 + e^{-i\omega}\right)
\end{aligned}$$

$$\begin{aligned}
z\text{-domain:} \quad A^{(1)}(z^2) &= \text{even powers in } 2H(z)A^{(0)}(z)H(z^{-1}) \\
&= \text{even powers in } \tfrac{1}{8}(z^2 + 4z + 6 + 4z^{-1} + z^{-2}) \\
A^{(1)}(z) &= \tfrac{1}{8}(z + 6 + z^{-1}).
\end{aligned}$$

Summary: The main point is to connect L^2 convergence of the infinite product for $\widehat{\phi}(\omega)$ with the number p of vanishing moments of the wavelets, and the smoothness s of $\phi(t)$. Everything depends on the eigenvalues of the matrix $\boldsymbol{T}_{2N-1}$:

Cascade convergence requires all $|\lambda| < 1$ except for a simple $\lambda = 1$.

Approximation of order p requires eigenvalues $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \ldots, (\frac{1}{2})^{2p-1}$.

Smoothness (s derivatives in L^2) requires all other $|\lambda| < 4^{-s}$.

Continuity of the Scaling Function

Functions in L^2 have finite energy. They may or may not be continuous. Continuity is a "pointwise" property, not revealed by inner products and not automatic under Condition E. (Haar satisfies this condition.) We describe now the test for continuity of $\phi(t)$. In borderline cases it is not always easy to apply, because it involves *two matrices*.

Those matrices are $\boldsymbol{m}(0)$ and $\boldsymbol{m}(1)$. Remember from Section 6.3 that these are $N \times N$ submatrices of the double-shift lowpass matrix $\boldsymbol{M} = (\downarrow 2)2\boldsymbol{H}$. The columns of $\boldsymbol{m}(0)$ and $\boldsymbol{m}(1)$ add to 1. If $\boldsymbol{e} = [1 \ \ldots \ 1]$ is the all-ones row vector then $\boldsymbol{em}(0) = \boldsymbol{e}$ and $\boldsymbol{em}(1) = \boldsymbol{e}$. The dilation equation in vector form is on the interval $[0, 1)$:

$$\boldsymbol{\Phi}(t) = \boldsymbol{m}(0)\,\boldsymbol{\Phi}(2t) + \boldsymbol{m}(1)\,\boldsymbol{\Phi}(2t-1). \tag{7.66}$$

The vector $\boldsymbol{\Phi}(t) = [\phi(t)\ \phi(t+1)\ \cdots]^T$ stacks the N slices of $\phi(t)$. Because equation (7.66) has only two coefficients, it gives a simple recursion (Section 6.3). The first digit t_1 in $t = .t_1t_2t_3\cdots$ tells whether we use $\boldsymbol{m}(0)$ or $\boldsymbol{m}(1)$:

$$\boldsymbol{\Phi}(t) = \boldsymbol{m}(t_1)\,\boldsymbol{\Phi}(.t_2t_3\cdots). \tag{7.67}$$

The next 0–1 digit t_2 tells whether the next step uses $\boldsymbol{m}(0)$ or $\boldsymbol{m}(1)$:

$$\Phi(t) = \boldsymbol{m}(t_1)\boldsymbol{m}(t_2)\,\Phi(.t_3t_4\cdots). \tag{7.68}$$

The matrices $\boldsymbol{m}(0)$ and $\boldsymbol{m}(1)$ can come in any order (determined by the digits in t). A nearby point T will *begin* with the same digits. At some later point the digits will differ. If $T = .t_1t_2T_3T_4\cdots$ then

$$\Phi(t) - \Phi(T) = \boldsymbol{m}(t_1)\boldsymbol{m}(t_2)[\,\Phi(.t_3t_4\cdots) - \Phi(.T_3T_4\cdots)]. \tag{7.69}$$

To prove continuity is to show that $\Phi(t)$ is close to $\Phi(T)$ when the neighbors t and T share many digits $t_1t_2\cdots t_K$. This will be true if the product of $\boldsymbol{m}$'s in every order is small. Actually we work with matrices of order $N-1$, after removing $\lambda = 1$.

Theorem 7.12 *The scaling function $\phi(t)$ is continuous if all products of $\boldsymbol{m}_{N-1}(0)$ and $\boldsymbol{m}_{N-1}(1)$ approach zero as the number of factors increases.*

The matrices $\boldsymbol{m}(0)$ and $\boldsymbol{m}(1)$ have the eigenvalue 1, with the all-ones left eigenvector $\boldsymbol{e}$. All products of the $\boldsymbol{m}$'s will have this eigenvalue and eigenvector. They won't go to zero! But in equation (7.69) these products multiply a vector that is *orthogonal* to $\boldsymbol{e}$:

$$\boldsymbol{e}\,\Phi(t) \equiv 1 \text{ implies that } \boldsymbol{e}[\,\Phi(.t_3t_4\cdots) - \Phi(.T_3T_4\cdots)] = 1 - 1 = 0. \tag{7.70}$$

Restricted to vectors orthogonal to $\boldsymbol{e}$, $\boldsymbol{m}(0)$ becomes $\boldsymbol{m}_{N-1}(0)$ and $\boldsymbol{m}(1)$ becomes $\boldsymbol{m}_{N-1}(1)$. If long products of these matrices are small, then (7.69) says that $\Phi(t)$ is close to $\Phi(T)$. This means that $\Phi(t)$ is continuous.

Continuity requires longer and longer products of $A = \boldsymbol{m}_{N-1}(0)$ and $B = \boldsymbol{m}_{N-1}(1)$ to approach zero. This may be easy to test, or hard. Any eigenvalue with $|\lambda| \geq 1$ guarantees failure. To have $\|A\| < 1$ and $\|B\| < 1$ guarantees success. But the right norm can be very difficult to find. The problem is the possibility of small eigenvalues and dangerous products:

$$A = \begin{bmatrix} \epsilon & 1 \\ 0 & 0 \end{bmatrix} \quad\text{and}\quad B = \begin{bmatrix} \epsilon & 0 \\ 1 & 0 \end{bmatrix} \quad\text{give}\quad AB = \begin{bmatrix} 1+\epsilon^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

The powers A^n and B^n go to zero. But $ABABAB\cdots$ blows up because $1+\epsilon^2 > 1$.

Problem 2 shows how to compute $\boldsymbol{m}_{N-1}$ from $\boldsymbol{m}_N$. We cannot show how to test all products in all orders. [Heil-Colella] and many others discuss this problem but it has no complete solution.

Smoothness of Binary Filters

A striking example of the difference between pointwise smoothness and derivatives in L^2 is the filter $\boldsymbol{h} = [-1\ 2\ 6\ 2\ -1]/8$. Its scaling function is infinite at all dyadic points. Pointwise, $\phi(t)$ is a failure. The matrix $\boldsymbol{M} = (\downarrow 2)2\boldsymbol{H}$ has a double eigenvalue at $\lambda = 1$, with only one eigenvector. The powers of $\boldsymbol{M}$ are therefore unbounded. But the eigenvalues of $\boldsymbol{T}$ — its MATLAB construction in Section 6.5 is followed by ***eig(T)*** — are $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0.5428, \ldots$. The smoothness of $\phi(t)$ is $s_{\max} = -\log(.5428)/\log 4 = .44$. Thus $\phi(t)$ and $w(t)$ have 0.44 derivatives with finite energy, in spite of the fact that $\phi(t)$ blows up on a dense set!

The maxflat Daubechies filters of length $4p-1$ have $2p$ zeros at π. Their scaling functions interpolate at the integers because D_{2p} is halfband: $\phi(n) = \delta(n)$ is the eigenvector of $\boldsymbol{M}$ for $\lambda = 1$. We expect high smoothness for $\phi(t)$ because of all the zeros:

$$2p = 0, 2, 4, 6 \quad \text{leads to} \quad s_{\max} = -0.5, 1.5, 2.44, 3.17.$$

We were also interested in the smoothness of the new binary filters (dual to these symmetric Daubechies halfband filters). A new 9/7 pair was constructed by lifting in Section 6.5. There we also balanced its zeros to 10/6 by moving $1+z^{-1}$ between analysis and synthesis. (This just changes $s_{\max}$ by 1.) The new filters were

$$\boldsymbol{h9} = [1 \;\; 0 \;\; -8 \;\; 16 \;\; 46 \;\; 16 \;\; -8 \;\; 0 \;\; 1]/64 \quad \text{with 2 zeros and} \quad s_{\max} = 0.59$$

$$\boldsymbol{h13} = [-1 \;\; 0 \;\; 18 \;\; -16 \;\; -63 \;\; 144 \;\; 348 \;\; \cdots]/512 \quad \text{with 4 zeros and} \quad s_{\max} = 1.18.$$

Compare with the standard symmetric biorthogonal FBI 9/7 pair, which has 16 nonzero coefficients. Two digits are inadequate but here they are:

$$\boldsymbol{h}\mathbf{FBI} = [0.03 \;\; -0.02 \;\; -0.08 \;\; 0.27 \;\; 0.60 \;\; \cdots] \text{ and } \boldsymbol{f}\mathbf{FBI} = [-0.05 \;\; -0.03 \;\; 0.30 \;\; 0.56 \;\; \cdots].$$

These have $s_{\max} = 1.4$ in analysis and 2.1 in synthesis. The FBI pair has higher coding gain and 4/4 zeros but no interpolating property. It gives higher PSNR and lower error on Lena and Barbara. But the perceptual quality of the new pair seems sharper (to our eyes). The analysis function $\widetilde{\phi}_{\text{new}}(t)$ is more peaked and the synthesis $\phi_{\text{new}}(t)$ is smoother. Our latest test on the "*boats*" image at 0.32 bpp was a tie in objective measures (PSNR, MSE, Max error), but we see more in the new image (Figure 7.4). The cable at the upper right is lost by the standard 9/7 pair, and the ship name PICARDY becomes unreadable. You see that the reality of filter comparison is not totally precise!

Figure 7.4: Original of *boats* and two competing 9/7 reconstructions.

Problem Set 7.3

1. Suppose $eA = e$. If x is a vector perpendicular to e, show that Ax is perpendicular to e. (If $ex = 0$ prove that $e(Ax) = 0$.)

2. Suppose $eA = e$. Show that multiplying $S^{-1}AS$ by blocks gives

$$\begin{bmatrix} I_{N-1} & 0_{N-1} \\ e_{N-1} & 1 \end{bmatrix}\begin{bmatrix} a_{N-1} & b_{N-1} \\ c_{N-1} & d \end{bmatrix}\begin{bmatrix} I_{N-1} & 0_{N-1} \\ -e_{N-1} & 1 \end{bmatrix} = \begin{bmatrix} A_{N-1} & b_{N-1} \\ 0_{N-1} & 1 \end{bmatrix}.$$

The first $N-1$ columns of S are perpendicular to e. The matrix $A_{N-1} = a_{N-1} - b_{N-1}e_{N-1}$ is the restriction of A to those vectors. Compute this restriction $m_{N-1}(0)$ for the matrix $m(0)$ from the hat coefficients $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

3. For the transform $\widehat{\phi}(\omega) = (1-e^{-i\omega})^2/i^2\omega^2$ of the hat function show that $\int |\omega|^{2s}|\widehat{\phi}(\omega)|^2\,d\omega < \infty$ if and only if $s < \frac{3}{2}$.

4. Find by hand the matrix T and its eigenvalues and $s_{\max}$, starting from the filters $h = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ and $h = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$.

5. Find by MATLAB the matrix T and its eigenvalues and $s_{\max}$ for the "D_6 filter" that has $h * h^T = \left(\frac{1+z^{-1}}{2}\right)^6 Q(z) =$ halfband of degree 10.

6. Show from their lengths that h and f and $p = h * f$ cannot all be symmetric halfband.

7. Dual to $f = [-1\ \ 0\ \ 9\ \ 16\ \ 9\ \ 0\ \ -1]/32$ is another *halfband but unsymmetric* filter $h^{\#} = [1\ \ 0\ \ 23\ \ 16\ \ -9\ \ 0\ \ 1]/32$. Verify that $p^{\#} = f * h^{\#}$ is halfband to give PR. This is our first unsymmetric product filter. What is the system delay l? What is $\lambda_{\max}(T^{\#})$ coming from $h^{\#}$? Why is $\widetilde{\phi}^{\#}(t)$ interpolating (equal to δ at the integers)? How smooth is it?

8. Explain why the smoothness of the delta function is $s_{\max} = -\frac{1}{2}$.

7.4 Splines and Semiorthogonal Wavelets

Splines are piecewise polynomials, with a smooth fit between the pieces. They are older than wavelets. The "two-scale equation" or dilation equation was at first not particularly noticed. Now we will see that the numbers $h(k)$ are binomial coefficients, directly from Pascal's triangle.

For a cubic spline the coefficients are $1, 4, 6, 4, 1$ divided by 16. The transfer function is $H(z) = \left(1 + z^{-1}\right)^4/16$. All four zeros are at $z = -1$. The filter is lowpass, the spline is as smooth as possible, and it has the highest accuracy $p = 4$ that is possible with $N = 4$. Almost every formula in this book comes out neatly and explicitly for splines.

One application of splines is to ***interpolation***, when data points need to be connected by a smooth curve. To put one high-degree polynomial through all the points is very unwise. A small movement of a single point produces an extreme change in the polynomial (which oscillates violently between the interpolation points). It is much better to use short pieces of low-degree polynomials — often cubic splines.

A cubic spline has degree 3, for any number of interpolation points. It has ***two continuous derivatives***, at the points where two different cubics meet. Thus $\phi_+(t)$ and $\phi'_+(t)$ and $\phi''_+(t)$ on one side of the meeting point agree with $\phi_-(t)$ and $\phi'_-(t)$ and $\phi''_-(t)$ on the other side. When $t = 0$ is the meeting point, only the coefficient of t^3 can change. The curve looks smooth and its coefficients are easy to find from the data points — but not trivial. There is a system of linear equations to solve, because all data points influence all coefficients. To say this in another way, the spline that matches the data values $0, 0, 1, 0, 0, \ldots$ is not zero in the intervals between those values. It decays exponentially as $|t| \to \infty$ but this "cardinal spline" does not give the best basis for computations.

The good function is the "B-spline" with compact support. It is our scaling function $\phi(t)$. This function matches the data values 0, $\frac{1}{6}$, $\frac{4}{6}$, $\frac{1}{6}$, 0, at the integers. It has unit area $\int \phi(t)dt = 1$ and a smooth fit (two derivatives). The spline is nonzero on ***four intervals*** (Figure 7.5). Outside this range $\phi(t)$ is identically zero.

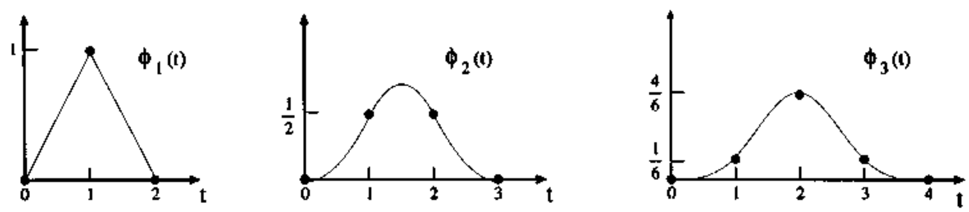

Figure 7.5: The spline $\phi_{N-1}(t)$ of degree $N-1$ is the convolution of N box functions. It is supported on $[0, N]$. Its filter $H(z) = (\frac{1+z^{-1}}{2})^N$ has a zero of order $p = N$ at $z = -1$.

A note on interpolation at the integers $t = n$. To match a set of values $f(n)$, we look for a combination of B-splines. The equations are $\sum F(k)\phi(n-k) = f(n)$. This is a constant-diagonal (Toeplitz) system, with $\frac{1}{6}$, $\frac{4}{6}$, $\frac{1}{6}$ on the diagonals coming from $\phi(n)$. To compute $F(k)$ we are inverting an FIR filter, and the inverse is IIR (a recursive filter). This explains why all $F(k)$ are nonzero when we interpolate an impulse $f(n) = \delta(n)$. It is like Shannon's Sampling Theorem, with cardinal splines instead of sinc functions. *In fact the splines converge to* $\frac{\sin \pi t}{\pi t}$ *as* $p = N \to \infty$.

One major advantage: Splines do not require equally spaced data points. At the limit of unequal spacing, pairs of nodes come together. The spline becomes a ***finite element*** with ***one*** continuous derivative. The cubic finite element with values 0, 1, 0, and slopes 0, 0, 0, is nonzero only on ***two intervals*** around the center point. A second cubic element interpolates 0, 0, 0, with slopes 0, 1, 0. This short support (Figure 7.7) makes finite elements very popular in solving differential equations — much more popular than splines.

When there are two data values (function and slope) at the mesh points, each input $x(n)$ to the corresponding filter is a *pair of numbers*. We have a ***multifilter*** and it leads to ***multiwavelets***. Everything is FIR. The values and slopes give the cubic in between. Multiwavelets are developed in the next section — we now return to the cubic spline $\phi(t)$.

Splines from Box Functions

Before two-scale equations, there was a direct approach to splines. This is still the fastest way. ***The cubic B-spline is the convolution of four box functions:***

$$\phi_3(t) = (B * B * B * B)(t). \tag{7.71}$$

The results from each convolution step are in Figure 7.5. These are linear splines $\phi_1(t)$, quadratic splines $\phi_2(t)$, and cubic splines $\phi_3(t)$. They are in the continuity classes C^0, C^1, and C^2, since they have 0, 1, and 2 continuous derivatives. ***The box function is*** $\phi_0(t)$. The convolution of N box functions has degree $N-1$ in each piece, with $N-2$ continuous derivatives between pieces.

We want to show that the fourth derivative of a cubic spline is *a sequence of delta functions*. The coefficients of these delta functions are $1, -4, 6, -4, 1$. These are the jumps in the third derivative, and this binomial pattern applies for every N. The first derivative of the hat function has jumps $1, -2, 1$.

To take the fourth derivative of a cubic spline, or the Nth derivative of a convolution of N box functions, we can work in time or frequency. We do both. First is the pleasant computation of $\widehat{\phi}_{N-1}(\omega)$ from convolving N box functions:

$$\widehat{\phi}_{N-1}(\omega) = (\widehat{\phi}_0(\omega))^N = \left(\frac{1}{i\omega}\right)^N (1 - e^{-i\omega})^N. \tag{7.72}$$

The box function has $\widehat{\phi}_0(\omega) = \int_0^1 e^{-i\omega t}\,dt = \frac{1}{i\omega}(1 - e^{-i\omega})$. Convolution in t is multiplication in ω, so the convolution of N box functions has transform $[\widehat{\phi}_0(\omega)]^N$.

Theorem 7.13 *The convolution of N boxes is a piecewise polynomial $\phi_{N-1}(t)$ of degree $N-1$. The jumps in the $(N-1)$st derivative at $t = 0, 1, \ldots, N$ are the alternating binomial coefficients $(-1)^t\binom{N}{t}$.*

Proof in the frequency domain: Each derivative multiplies the transform by $i\omega$. The Nth derivative cancels the denominator in $\widehat{\phi}_{N-1}(\omega)$ and has transform $(1 - e^{-i\omega})^N$. This is the transform of a sequence of delta functions at the points $t = 0, 1, \ldots, N$. Since the Nth derivative is zero between those points, the spline $\widehat{\phi}_{N-1}(t)$ must be a piecewise polynomial of degree $N-1$.

The fourth derivative of $\phi_3(t)$ has transform $(1 - e^{-i\omega})^4$. So the third derivative has jumps $1, -4, 6, -4, 1$.

Proof in the time domain: The derivative of $f(t) * g(t)$ is $f'(t) * g(t)$ or equally it is $f(t) * g'(t)$. The fourth derivative of the cubic $\phi = B * B * B * B$ has four factors:

$$\frac{d^4\phi}{dt^4} = (B' * B' * B' * B')(t). \tag{7.73}$$

Each factor $B'(t)$ is $\delta(t) - \delta(t-1)$. This is the derivative of the box function, which jumps up at $t = 0$ and down at $t = 1$. The convolution (7.73) becomes

$$\frac{d^4\phi}{dt^4} = \delta(t) - 4\delta(t-1) + 6\delta(t-2) - 4\delta(t-3) + \delta(t-4). \tag{7.74}$$

The third derivative has jumps $1, -4, 6, -4, 1$ at $t = 0, 1, 2, 3, 4$. Otherwise $d^4\phi/dt^4 = 0$ and $\phi(t)$ is an ordinary cubic polynomial.

The Coefficients $h(n)$ for Splines

We know that the filter coefficients $\frac{1}{2}, \frac{1}{2}$ lead to the box function $\phi_0(t)$. The coefficients $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ lead to the hat function $\phi_1(t)$. These numbers appear in the lowpass filter $\boldsymbol{H}$. We suspect that the coefficients $\boldsymbol{h}(n) = \frac{1}{16}(1, 4, 6, 4, 1)$ lead to the cubic spline $\phi_3(t)$. The pattern for the convolution of N box functions could be described in two ways:

1. $\boldsymbol{h}(n)$ are the coefficients in the special polynomial $H(z) = \left(\frac{1+z^{-1}}{2}\right)^N$.

2. $\boldsymbol{h}(n)$ is the convolution $\left(\frac{1}{2}, \frac{1}{2}\right) * \left(\frac{1}{2}, \frac{1}{2}\right) * \cdots * \left(\frac{1}{2}, \frac{1}{2}\right)$.

Those two patterns are equivalent. Each multiplication by $(\frac{1+z^{-1}}{2})$ in the z-domain is convolution by $(\frac{1}{2}, \frac{1}{2})$ in the time domain. The question is, *what is the effect on the scaling function*? Apparently it is convolved with the box function.

This suggests a more general pattern. The transfer function $F(z)G(z)$ and the filter coefficients $f(n) * g(n)$ correspond to the scaling function $\phi_f(t) * \phi_g(t)$. *Multiplication of filters gives convolution of scaling functions!* This is not hard to prove. It applies immediately to the special case $F(z) = \frac{1+z^{-1}}{2}$. Convolution with the box function gives the next spline scaling function, one degree higher.

Lemma 7.2 *The scaling function $\phi_h(t)$ corresponding to $\boldsymbol{H} = \boldsymbol{FG}$ and to $\boldsymbol{h}(n) = \boldsymbol{f}(n) * \boldsymbol{g}(n)$ is the convolution of the scaling functions for $\boldsymbol{F}$ and $\boldsymbol{G}$:*

$$\phi_h(t) = \phi_f(t) * \phi_g(t) \ \text{ and } \ \widehat{\phi}_h(\omega) = \left(\widehat{\phi}_f(\omega)\right)\left(\widehat{\phi}_g(\omega)\right). \tag{7.75}$$

Proof The dilation equations for ϕ_f and ϕ_g involve ω and $\omega/2$:

$$\widehat{\phi}_f(\omega) = F(\tfrac{\omega}{2})\,\widehat{\phi}_f(\tfrac{\omega}{2}) \quad \text{and} \quad \widehat{\phi}_g(\omega) = G(\tfrac{\omega}{2})\,\widehat{\phi}_g(\tfrac{\omega}{2}).$$

Multiply to get $\widehat{\phi}_h(\omega) = H(\frac{\omega}{2})\widehat{\phi}_h(\frac{\omega}{2})$. This is the dilation equation for $\boldsymbol{H}$. If you like infinite products, multiply $\prod F(\omega/2^i)$ and $\prod G(\omega/2^i)$ to get $\widehat{\phi}_h(\omega) = \prod H(\omega/2^i)$.

Example 7.6. If the coefficients in $\boldsymbol{f}$ and $\boldsymbol{g}$ are both $\frac{1}{2}, \frac{1}{2}$, the scaling functions $\phi_f(t)$ and $\phi_g(t)$ are the box function. Then $\boldsymbol{h} = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ produces the hat function:

$$\begin{aligned}
(\tfrac{1}{2}, \tfrac{1}{2}) * (\tfrac{1}{2}, \tfrac{1}{2}) &= (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}) \text{ because } \left(\tfrac{1+z^{-1}}{2}\right)\left(\tfrac{1+z^{-1}}{2}\right) &&= \tfrac{1+2z^{-1}+z^{-2}}{4}.\\
B(t) * B(t) &= \text{Box function} * \text{Box function} &&= \text{Hat function.}
\end{aligned}$$

Every extra factor $(\frac{1+z^{-1}}{2})$ means a convolution with the box function. The new $\phi(t)$ has one more derivative. For splines, each new $\phi_N(t) = \phi_{N-1}(t) * B(t) = \int_0^1 \phi_{N-1}(t-x)\,dx$ is one degree higher. This gives a nice formula for the time derivative:

$$\phi_N'(t) = \int_0^1 \phi_{N-1}'(t-x)\,dx = \phi_{N-1}'(t) - \phi_{N-1}'(t-1). \tag{7.76}$$

The spline of degree $N-1$ is the convolution of N box functions, and the corresponding filter has $H(z) = (\frac{1+z^{-1}}{2})^N$. We can verify in one step that this spline satisfies the correct dilation equation! Its transform $\widehat{\phi}_{N-1}(\omega)$ agrees with $H\left(\frac{\omega}{2}\right)\widehat{\phi}_{N-1}\left(\frac{\omega}{2}\right)$:

$$\left(\frac{1-e^{-i\omega}}{i\omega}\right)^N = \left(\frac{1+e^{-i\omega/2}}{2}\right)^N\left(\frac{1-e^{-i\omega/2}}{i\omega/2}\right)^N. \tag{7.77}$$

We started with $\phi(t)$ and determined $\boldsymbol{h}(k)$. Normally we do the opposite — solve the dilation equation for $\phi(t)$. Here we have found the dilation equation for a spline:

$$\phi_{N-1}(t) = 2^{1-N}\sum_0^N \binom{N}{k}\phi_{N-1}(2t-k). \tag{7.78}$$

The lowpass space V_0 contains all smooth splines of degree $N-1$ on unit intervals. The B-splines are its basis (not orthogonal!). The space V_1 contains piecewise polynomials on *half*-intervals, and therefore contains V_0. The dilation equation (7.78) expresses $\phi_{N-1}(t)$ in V_0 as a combination of basis functions of V_1. It remains to find the other space W_0 which contains the wavelets.

Summary: $H(z)$ has a zero of order $p = N$ at $z = -1$. *There are N zeros at* $\omega = \pi$. Correspondingly, the polynomials $1, t, t^2, \ldots, t^{N-1}$ are combinations of the splines of degree $N - 1$. Those polynomials are all in the lowpass space V_0. The accuracy is $p = N$. The wavelets will have N vanishing moments. ***What are those spline wavelets?***

Inner Products and Riesz Bounds

The inner products of splines with their translates give the values of higher splines. The inner products for the hat function are $\boldsymbol{a} = \frac{1}{6}, \frac{4}{6}, \frac{1}{6}$:

$$\boldsymbol{a}(0) = 2\int_0^1 t^2\,dt = \tfrac{4}{6} \text{ and } \boldsymbol{a}(1) = \boldsymbol{a}(-1) = 2\int_0^1 t(t-1)\,dt = \tfrac{1}{6}.$$

Those numbers agree with the cubic B-spline at $t = 1, 2, 3$. The formula is:

$$\textit{Spline Inner Products} \quad \boldsymbol{a}(k) = \int_{-\infty}^{\infty} \phi_{N-1}(t)\,\phi_{N-1}(t+k)\,dt = \phi_{2N-1}(N+k). \tag{7.79}$$

The integral is convolving N box functions with N more box functions — and shifting by k. The $2N$ boxes produce the higher spline ϕ_{2N-1}.

The vector $\boldsymbol{a}$ of inner products solves $\boldsymbol{a} = \boldsymbol{T}\boldsymbol{a}$. The operator $\boldsymbol{T}$ comes from the product $H(z)H(z^{-1})$, which for splines is $(\frac{1+z^{-1}}{2})^N(\frac{1+z}{2})^N$. This is z^N times the function $(\frac{1+z^{-1}}{2})^{2N}$ that gives the higher spline $\phi_{2N-1}(t)$. In other words, *the matrix* $\boldsymbol{T}$ *for the lower spline is identical to the matrix* $\boldsymbol{M}$ *for the higher spline.*

The eigenvector of $\boldsymbol{M}$ gives the values of the higher spline at the integers. The same eigenvector (of $\boldsymbol{T}$!) gives the inner products of the lower splines in (7.79). All inner products have $\boldsymbol{a}(k) \geq 0$, because all splines have $\phi(t) \geq 0$. The inner products sum to 1. The maximum value of $A(\omega) = \sum \boldsymbol{a}(k)e^{-ik\omega}$ is $B = 1$, which occurs at $\omega = 0$. It is not hard to show that the minimum value of $A(\omega)$ occurs at $\omega = \pi$:

$$A_{\min} = \sum (-1)^k \phi_{2N-1}(N+k). \tag{7.80}$$

Thus the translates $\phi_{N-1}(t-k)$ form a Riesz basis for V_0 with bounds $A_{\min}$ and 1. For the hat function ($N = 2$) the lower bound is $A_{\min} = -\frac{1}{6} + \frac{4}{6} - \frac{1}{6} = \frac{1}{3}$. In frequency this means that

$$\tfrac{1}{3} \leq A(\omega) = \sum_{-\infty}^{\infty} \left|\widehat{\phi}_1(\omega + 2\pi k)\right|^2 = \tfrac{4}{6} + \tfrac{2}{6}\cos\omega \leq 1. \tag{7.81}$$

The basis of hat functions is well conditioned. Of course it is not orthogonal. If we orthogonalize, following the shift-invariant method of Section 6.5, we get the Battle-Lemarié $\phi(t)$ and $w(t)$ — smooth, quickly decaying, infinite support.

Spline Wavelets

There are several important possibilities for the wavelets. One family is FIR and biorthogonal, the other is IIR and "semiorthogonal." Semiorthogonal wavelets are perpendicular to the spline $\phi(t)$, but they are not orthogonal among themselves.

The FIR biorthogonal construction follows the usual rules. We need a halfband product filter $P(z)$. One factor is $(\frac{1+z^{-1}}{2})^N$, to produce the spline. This can be $H(z)$ in the analysis bank, or (better) it can be $F(z)$ in the synthesis bank. The other bank must contain an extra factor to make the product halfband. The natural choices for this second filter are $(\frac{1+z^{-1}}{2})^{2p-N}Q(z)$, which brings the product filter back to the standard Daubechies polynomial (where $Q(z)$ has degree $2p-2$). We need $2p > N$ to have a zero at $z=-1$. For the smallest p, the second filter may not satisfy Condition E for a stable basis in L^2; *more zeros may be needed.* The low degree spline filters and dual filters are the best known:

$$F(z) = (\tfrac{1+z^{-1}}{2})^2 \quad \text{goes with} \quad H(z) = (\tfrac{1+z^{-1}}{2})^2(-1+4z^{-1}-z^{-2}) \;:\; \textit{stable}$$

$$F(z) = (\tfrac{1+z^{-1}}{2})^3 \quad \text{goes with} \quad H(z) = (\tfrac{1+z^{-1}}{2})(-1+4z^{-1}-z^{-2}) \;:\; \textit{unstable}$$

$$F(z) = (\tfrac{1+z^{-1}}{2})^4 \quad \text{goes with} \quad H(z) = (\tfrac{1+z^{-1}}{2})^4 Q_6(z) \;:\; \textit{stable}.$$

The new book [CR] is a very good reference for biorthogonal FIR filters.

Example 7.7. Hat function from $F(z) = (\frac{1+z^{-1}}{2})^2$ gives the 5/3 filter bank.

If the analysis filter has $H(z) \equiv 1$, the product with $F(z)$ is halfband. The analysis scaling function $\widetilde{\phi}(t)$ is the delta function. Its translates $\delta(t-k)$ are biorthogonal to the hat functions $H(t-k)$. But the delta function is not acceptable.

If $H(z)$ is given two zeros at π, it needs the extra $Q(z)$:

$$H(z) = \left(\frac{1+z^{-1}}{2}\right)^2\left(\frac{-1+4z^{-1}-z^{-2}}{2}\right) = \frac{-1+2z^{-1}+6z^{-2}+2z^{-3}-z^{-4}}{8}.$$

Those coefficients $-1, 2, 6, 2, -1$ appeared in the Guide to the Book. This is the analysis part of the biorthogonal $5/3$ pair. The synthesis part is the linear spline (the hat). The product $P(z) = F(z)H(z)$ has four zeros at $z=-1$. Instead of the orthogonal D_4 factors of $P(z)$, the spline $5/3$ factorization gives linear phase.

In these biorthogonal examples, one scaling function is a spline. Best if this is ***synthesis***. The other scaling function is not a spline or a combination of splines or a piecewise polynomial. The space $\widetilde{V}_0$ spanned by $\widetilde{\phi}(t-k)$ is different from V_0. This is normal, but spline people expect to live and work exclusively in spline spaces. They want $V_0 = \widetilde{V}_0 =$ all splines of degree $N-1$. We now achieve this, but an IIR filter appears in the analysis half of the filter bank.

Semiorthogonal Wavelets

Start with a basis $\{\phi(t-k)\}$ for V_0. Suppose this basis is not orthogonal. The hat function and the cubic B-spline are examples of $\phi(t)$. We want to find wavelets $w(t-k)$ that are *orthogonal* to $\phi(t)$. Thus we maintain what was true in the fully orthogonal case:

$$V_0 \perp W_0 \quad \text{and} \quad V_0 \oplus W_0 = V_1. \tag{7.82}$$

The spaces are orthogonal but the bases within those spaces are not orthogonal.

Multiresolution in the biorthogonal case always has

$$V_0 \perp \widetilde{W}_0 \text{ and } W_0 \perp \widetilde{V}_0 \text{ and } V_0 + W_0 = V_1 \text{ and } \widetilde{V}_0 + \widetilde{W}_0 = \widetilde{V}_1. \tag{7.83}$$

Compare with (7.82) to see that $V_0 = \widetilde{V}_0$ and $W_0 = \widetilde{W}_0$. At every scale we will have $V_j \perp W_j$ and $V_j = \widetilde{V}_j$ and $W_j = \widetilde{W}_j$. There is only *one* multiresolution in the semiorthogonal case, one family $V_0 \subset V_1 \subset V_2$, as in the orthogonal case. The difference is that we have two bases for V_0, the given basis $\phi(t-k)$ and the biorthogonal basis $\widetilde{\phi}(t-k)$. This applies at every scale j. The tilde is needed for the dual basis, even if it is not needed for the space.

Semiorthogonality has an important property, directly from (7.83):

Semiorthogonal wavelets $w\left(2^j t - k\right)$ *and* $w\left(2^J t - l\right)$ *are perpendicular if* $j \neq J$.

At the same scale $j = J$, semiorthogonal wavelets are not generally perpendicular. But because W_j is orthogonal to V_j which contains all previous $W_{j-1}, W_{j-2}, \ldots$, we are guaranteed that W_j is perpendicular to all wavelets at other scales.

To construct this new wavelet $w(t)$, we need the highpass coefficients $f_1(k)$. In the orthogonal case, they come from $f_0(k)$ by an alternating flip. $F_1(z)$ is $-z^{-N} F_0\left(-z^{-1}\right)$, where z^{-1} gives the flip and $-z^{-1}$ makes it alternating. In that orthogonal case, the inner products $\langle\phi(t), \phi(t+k)\rangle$ are $a(k) = \delta(k)$. The polynomial $A(z) = \sum a(k)z^{-k}$ is identically 1. In the semiorthogonal case, when $\{\phi(t-k)\}$ is not orthonormal, *this inner product polynomial enters the highpass coefficients.*

The highpass function $F_1(z)$ becomes an alternating flip of $F_0(z)A(z)$. The analysis filters become IIR. Here is the general rule for semiorthogonality:

Theorem 7.14 *Suppose the lowpass* $F_0(z)$ *leads to scaling functions* $\phi(t-k)$ *whose inner products are the coefficients in* $A(z)$. *Then the highpass* $F_1(z)$ *that yields semiorthogonal wavelets* $w(t-k)$ *is the alternating flip of* $F_0(z)A(z)$:

$$F_1(z) = -z^{1-2N} F_0\left(-z^{-1}\right) A\left(-z^{-1}\right). \tag{7.84}$$

Proof. We want $\int \phi(t) w(t-n)\,dt = 0$ for all n. Use the dilation equation for $\phi(t)$ and the wavelet equation for $w(t)$:

$$\int_{-\infty}^{\infty} \left[\sum 2f_0(k)\phi(2t-k)\right]\left[\sum 2f_1(k)\phi(2t-2n-k)\right] dt = 0. \tag{7.85}$$

Change the second sum to $\sum 2f_1(\ell-2n)\phi(2t-\ell)$. The inner product of $\phi(2t-k)$ with $\phi(2t-\ell)$ is $a(\ell-k)$. The orthogonality requirement (7.89) becomes

$$\sum_{\ell}\sum_{k} f_0(k)a(\ell-k)f_1(\ell-2n) = 0. \tag{7.86}$$

The highpass filter is double-shift orthogonal, but not to the lowpass filter. The double-shift orthogonality is to the sequence $\sum f_0(k)a(\ell-k)$ which corresponds to $F_0(z)A(z)$. Therefore $F_1(z)$ comes from $F_0(z)A(z)$ by an alternating flip. This completes the proof.

The sequence $f_0(0), \ldots, f_0(N)$ gives a scaling function supported on $[0, N]$. The inner product $a(N) = \int \phi(t)\phi(t+N)\,dt = a(-N)$ is automatically zero. At most, the symmetric polynomial $A(z)$ has terms $a(k)z^{-k}$ and $a(k)z^{k}$ for $k < N$. Note that $A(z) = A\left(z^{-1}\right)$. The degree of $F_0(z)A(z)$ is at most $2N-1$.

Example 7.8. The hat function has $a(0) = \frac{4}{6}$ and $a(1) = a(-1) = \frac{1}{6}$. Find $F_1(z)$.

The product $F_0(z)A(z)$ is $\frac{1}{24}\left(1+2z^{-1}+z^{-2}\right)\left(z+4+z^{-1}\right)$. By alternating flip

$$\begin{aligned} F_1(z) &= \frac{-z^{-3}}{24}\left(1-2z+z^2\right)\left(-z+4-z^{-1}\right) \\ &= \tfrac{1}{24}\left(1-6z^{-1}+10z^{-2}-6z^{-3}+z^{-4}\right). \end{aligned}$$

The highpass coefficients $2f_1(k)$ yield $w(t)$ as drawn in Figure 7.6:

$$w(t) = \tfrac{1}{12}\left[\phi(2t) - 6\phi(2t-1) + 10\phi(2t-2) - 6\phi(2t-3) + \phi(2t-4)\right]. \tag{7.87}$$

This is orthogonal to all hat functions $\phi(t-k)$. It is *not* orthogonal to all $w(t-k)$. But it *is* orthogonal to every $w\left(2^j t - k\right)$ for $j \neq 0$. That is semiorthogonality.

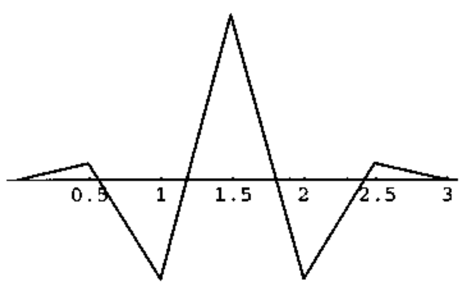

Figure 7.6: The linear wavelet $w(t)$ orthogonal to the hat functions $\phi(t-k)$.

The IIR Half of the Semiorthogonal Filter Bank

The analysis functions $H_0(z)$ and $H_1(z)$ must come from the perfect reconstruction condition $\boldsymbol{H}_m(z)\boldsymbol{F}_m(z) = 2z^{-\ell}\boldsymbol{I}$. We know the modulation matrix $\boldsymbol{F}_m(z)$. Its inverse times $2z^{-\ell}$ gives $\boldsymbol{H}_m(z)$:

$$\begin{bmatrix} F_0(z) & F_1(z) \\ F_0(-z) & F_1(-z) \end{bmatrix} = \begin{bmatrix} F_0(z) & -z^{1-2N}F_0\left(-z^{-1}\right)A\left(-z^{-1}\right) \\ F_0(-z) & z^{1-2N}F_0\left(z^{-1}\right)A\left(z^{-1}\right) \end{bmatrix}$$

$$\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = \frac{2z^{-\ell}}{D(z)}\begin{bmatrix} z^{1-2N}F_0\left(-z^{-1}\right)A\left(z^{-1}\right) & z^{1-2N}F_0\left(-z^{-1}\right)A\left(-z^{-1}\right) \\ -F_0(-z) & F_0(z) \end{bmatrix}$$

The key is always in the determinant $D(z)$. For an FIR filter bank, $D(z)$ is the delay $z^{-\ell}$. For these semiorthogonal filters, we don't get a delay but we do get a remarkable formula:

$$\begin{aligned} D(z) &= z^{1-2N}\left[F_0(z)F_0\left(z^{-1}\right)A\left(z^{-1}\right) + F_0(-z)F_0\left(-z^{-1}\right)A\left(-z^{-1}\right)\right] \\ &= z^{1-2N}A\left(z^{-2}\right). \end{aligned} \tag{7.88}$$

This is the equation $\boldsymbol{Ta} = \boldsymbol{a}$ for the eigenvector of the transition matrix! The matrix $\boldsymbol{T}$ in the time domain is $(\downarrow 2)2\boldsymbol{F}\boldsymbol{F}^T$. In the z-domain this is exactly what appears in (7.88); the aliasing term with $-z$ comes from $(\downarrow 2)$. The determinant $D(z)$ is identified. It is an odd function, and we take $\ell = 2N-1$. The analysis filters can be read from the matrix $\boldsymbol{H}_m(z)$:

$$H_0(z) = \frac{2z^{-\ell}F_0\left(z^{-1}\right)A\left(z^{-1}\right)}{A\left(z^{-2}\right)} \quad \text{and} \quad H_1(z) = \frac{2F_0(-z)}{A\left(z^{-2}\right)}. \tag{7.89}$$

The division by $A\left(z^{-2}\right)$ yields IIR filters except in the orthogonal case when $A \equiv 1$.

Example 7.9. (Hats continued) The modulation matrix from F_0 and F_1 is

$$\frac{1}{4}\cdot\frac{1}{24}\begin{bmatrix} 1+2z^{-1}+z^{-2} & 1-6z^{-1}+10z^{-2}-6z^{-3}+z^{-4} \\ 1-2z^{-1}+z^{-2} & 1+6z^{-1}+10z^{-2}+6z^{-3}+z^{-4} \end{bmatrix}.$$

The determinant is verified to be $\frac{1}{6}\left(z^{-1}+4z^{-3}+z^{-5}\right) = z^{-3}A\left(z^{-2}\right)$.

The synthesis coefficients require a division by $A\left(z^{-2}\right)$. This has zeros at $z^2 = 2 \pm \sqrt{3}$. We can express $1/A\left(z^{-2}\right)$ by partial fractions (Problem 7). The power series for these fractions will have $\left(2-\sqrt{3}\right)^n$ in the coefficients of z^{2n} and z^{-2n}. That gives the decay rate of the (IIR!) filter coefficients $\boldsymbol{h}(k)$. The scaling functions $\tilde{\phi}(t-k)$ are biorthogonal to the hats.

Summary: Splines open the possibility of maintaining $V_j \perp W_j$ in the biorthogonal case. Then the wavelets are semiorthogonal. The highpass coefficients involve the inner products $\boldsymbol{a}(k)$ of the scaling functions. The associated polynomial $A(z)$ is called the "*Euler-Frobenius polynomial*" in spline theory, but there is no restriction to splines. For every lowpass filter that yields a scaling function $\phi(t)$, the theory produces semiorthogonal wavelets.

For splines of degree $p-1$, the inner products $\boldsymbol{a}(k)$ are the values at integers of the B-splines of degree $2p-1$. For all cases, the inner products are in the eigenvector $\boldsymbol{Ta} = \boldsymbol{a}$.

We emphasize that the special properties of splines are attracting a lot of attention in signal processing. They have maximum regularity (and symmetry) with minimum support and complexity. ***Splines are outstanding in synthesis.*** They give approximation of high order p with low constant C in the error $C(\Delta t)^p$. The dual analysis filter has to be longer, but no construction will ever be perfect.

Problem Set 7.4

1. Prove that the spline $\phi_{N-1}(t)$ is symmetric about the center point $t = N/2$.
2. Explain the formula $\phi_{N-1}(t) = \frac{1}{(N-1)!}\sum(-1)^k\binom{N}{k}(t-k)_+^{N-1}$. Here $(t-k)_+ = \max(t-k, 0)$. One proof uses the jumps in the $(N-1)$st derivative.
3. Take derivatives in Problem 2 to verify $\phi_N'(t) = \phi_{N-1}(t) - \phi_{N-1}(t-1)$.
4. (Challenge) Prove that $\phi_N(t) = \frac{t}{N}\phi_{N-1}(t) + \frac{N+1-t}{N}\phi_{N-1}(t-1)$. This recursion gives a quick stable computation of $\phi_N(t)$. Hint: take out a factor to get $\phi_N(t) = \frac{t}{N}\phi_N'(t) + \frac{N+1}{N}\phi_{N-1}(t-1)$. Use Problems 2–3.
5. Suppose $P(z) = F(z)H(z)$ is centered halfband. Why is $\phi_f(t) * \phi_h(t)$ equal to $\delta(n)$ at $t = n$?
6. Find the polyphase matrix for $\boldsymbol{h}_0(z) = \frac{1}{4}(1, 2, 1)$ and $\boldsymbol{h}_1(z) = \frac{1}{24}(1, -6, 10, -6, 1)$. Connect its determinant to $A(z)$.

7. Find A and B in the partial fraction expansion

$$\frac{6}{x^2+4x+1} = \frac{A}{x+2-\sqrt{3}} + \frac{B}{x+2+\sqrt{3}}.$$

Expand the last two fractions in powers of $\frac{2-\sqrt{3}}{x}$ and $\frac{x}{2+\sqrt{3}}$. Combine into a power series for $6/(x+4+x^{-1})$. The coefficients of x^n and x^{-n} should be equal.

8. Show that the inner products $a(k) = \int \phi(t)\phi(t+k)\,dt$ for quadratic splines are $a = \frac{1}{120}(1, 26, 66, 26, 1)$. You could verify that this is the eigenvector in $Ta = a$, or evaluate the 5th-degree spline at the integers. What is the support interval of the semiorthogonal quadratic $w(t)$?

9. For splines show that $A(\omega) = \sum |\widehat{\phi}_{N-1}(\omega+2\pi k)|^2$ equals $\left(2\sin\frac{\omega}{2}\right)^{2N} \sum(\omega+2\pi k)^{-2N}$. Verify $A'(\pi) = 0$. The minimum is at $\omega = \pi$.

7.5 Multifilters and Multiwavelets

This brief section describes a recent development — to allow the filter coefficients $h(k)$ to be $r \times r$ matrices. Each input sample $x(n)$ is a vector with r components. So is each output $\widehat{x}(n)$. The bank of multifilters has the same structure as an ordinary filter bank, with the extra freedom that comes with matrix coefficients.

In continuous time, the dilation equation will have matrix coefficients $h(k)$. The solution gives r scaling functions $\phi_1(t), \ldots, \phi_r(t)$. Then the wavelet equation with highpass matrices $h_1(k)$ yields r corresponding wavelets. *Properly chosen, all these functions can have symmetry as well as orthogonality!* They and all their translates are orthogonal when the polyphase matrix (of order $2r$) is paraunitary. Multiresolution produces an orthonormal basis of wavelets — the translates of r functions at all scales $-\infty < j < \infty$:

$$w_{ijk}(t) = 2^{-j/2} w_i\left(2^j t - k\right), \quad 1 \le i \le r.$$

In some situations $r > 1$ is quite reasonable. When sampling a function $x(t)$, we may also sample its slope $x'(t)$. Velocity may be involved as well as displacement. The pair $\left(x(n), x'(n)\right)$ is a vector with $r = 2$. Those samples could go through two separate scalar filter banks, or through one bank of multifilters. This example already shows, because $x'(t)$ is dimensionally different from $x(t)$, that scaling is important for the r inputs.

The cubic scaling functions in this $\left(x(n), x'(n)\right)$ example are *"finite elements."* They are drawn in Figure 7.7, with support $[0, 2]$. Like splines, these cubics $\phi_1(t)$ and $\phi_2(t)$ have linear phase — but they are not orthogonal to their translates. They have *one* continuous derivative, not two. The space of C^1 cubics has a more local basis (but with $r = 2$ functions per interval), while the spline space of C^2 cubics has one basis function per interval (the B-spline).

Figure 7.7 also displays the wavelets $w_1(t)$ and $w_2(t)$ in the semiorthogonal case. They are supported on $[0, 3]$. They and their translates span W_0 and are orthogonal to $\phi_1(t)$ and $\phi_2(t)$. They are piecewise cubic on half-intervals, and they come from a wavelet equation $w(t) = \sum 2h_1(k)\phi(2t-k)$. Those coefficients $h_1(k)$ are 2×2 matrices!

The finite element spaces V_0 of degree $1, 3, 5, 7$ are spanned by $r = 1, 2, 3, 4$ scaling functions. Degree 1 has $\phi(t) =$ hat function which has C^0 smoothness (no continuous derivatives). From the polynomials contained in V_0 we know the accuracy $p = 2, 4, 6, 8$. The smoothness is C^0, C^1, C^2, C^3 (and the pattern continues). The r semiorthogonal multiwavelets are always

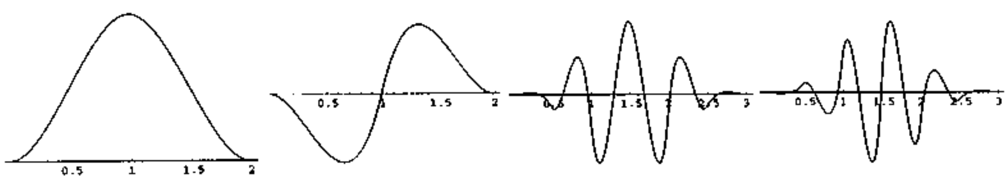

Figure 7.7: Cubic multiwavelets $w_1(t)$ and $w_2(t)$ orthogonal to cubic finite elements $\phi_1(t-k)$ and $\phi_2(t-k)$.

supported on $[0, 3]$. These finite element scaling functions are cousins to the splines — which have more smoothness but longer support. Here is a piecewise linear construction with even less smoothness and even shorter support.

Multiwavelet example: "Haar's Hat" These functions are *linear* on each interval but *not continuous* at the ends. Each piece is a straight line between the end values. That piece is determined by $r = 2$ numbers, its average and its slope. Its average goes with the box function $\phi_1(t)$, its slope goes with $\phi_2(t) = 2t - 1$. Those scaling functions on $[0, 1]$ are combinations of the same functions on half-intervals:

Box: $\phi_1(t) = \phi_1(2t) + \phi_1(2t-1)$ (***usual Haar equation***)

Slope: $\phi_2(t) = \frac{1}{2}[\phi_2(2t) + \phi_2(2t-1) - \phi_1(2t) + \phi_1(2t-1)]$ (***zero outside*** $[0, 1]$)

This is a ***matrix dilation equation***. The coefficients are 2×2 matrices $\boldsymbol{h}(0)$ and $\boldsymbol{h}(1)$:

$$\begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_1(2t) \\ \phi_2(2t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_1(2t-1) \\ \phi_2(2t-1) \end{bmatrix} \tag{7.90}$$

You see the eigenvalues 1 and $\frac{1}{2}$ that always accompany linear functions in V_0. The accuracy is $p = 2$. The support is short! This is a block transform, not overlapping the next pair of samples — the average and slope on the next interval $[1, 2]$. We get good accuracy, but the price is complete lack of smoothness.

It is an exercise to find the wavelets for Haar's Hat. They will be linear on half-intervals (and still discontinuous). Another example of two scaling functions: the real and imaginary parts of a complex scaling function. Symmetry with orthogonality is possible [Lena]. We turn to a more magical construction that combines orthogonality and symmetry and short support and continuity.

This special construction by Geronimo, Hardin, and Massopust gave a strong impetus to multiwavelet theory. The functions $\phi_1(t)$ and $\phi_2(t)$ in Figure 7.8 were found by a recursive interpolation process. The 2×2 coefficients $\boldsymbol{h}_0(k)$ in their dilation equation came later. So did the highpass coefficients and the wavelets [StSt1, GHM2]. All these functions have linear phase and short support and orthogonality. The only earlier example with these properties was Haar's. Now the accuracy is $p = 2$, because the scaling functions can reproduce a hat:

$$\phi_1(t) + \phi_1(t-1) + \phi_2(t) = \text{hat function.}$$

The space V_0 *has an FIR orthogonal basis* $\{\phi_1(t-k)\}$ *joined with* $\{\phi_2(t-k)\}$:

$$f_0(t) = \sum a_{10k}\,\phi_1(t-k) + a_{20k}\,\phi_2(t-k) \ \text{ is in } V_0. \tag{7.91}$$

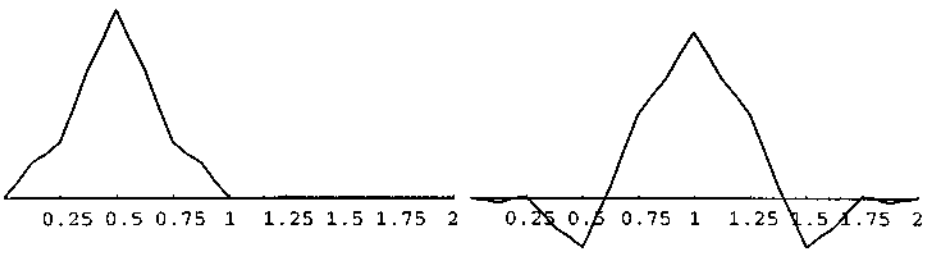

Figure 7.8: GHM scaling functions combine orthogonality of all translates and linear phase.

There is a serious problem for multifilters and multiwavelets. Everyone in signal processing asks about it immediately. We have twice as many filters (or r times as many filters) as usual. One input stream $x(n)$ produces $2r$ half-length outputs from the analysis bank. This means extra computation (but shorter filters). It also means that the stream $x(n)$ has to be *vectorized*—so r inputs go together. It is not at all satisfactory just to take the original stream in blocks of r samples. This does violence to the whole idea. The $\phi_i(t)$ are not time shifts of one function, they are r functions *at each time*.

The $\big(x(n), x'(n)\big)$ example comes naturally in blocks of two at each time. So does Haar's Hat. For other multifilters, one possibility is to repeat $(x(n), cx(n))$ with a suitable scalar c. What we really want is a discrete form of (7.91). The recent papers [Heller2, XiaG] show how to convert a single input stream to multi-inputs which give samples of the data at half-intervals (thus $r = 2$ samples per interval). These Geronimo-Hardin-Massopust multiwavelets have given the first experimental results in compression. In competition against D_4 wavelets, with the same accuracy $p = 2$, the multiwavelets gave better compression and required more computations.

Perfect Reconstruction and Orthogonality and Accuracy

The theory of multiwavelets looks completely familiar up to one point, where everything changes. The multifilters still have transfer functions $\boldsymbol{H}(z) = \sum \boldsymbol{h}(k)z^{-k}$. These are now matrices. The perfect reconstruction conditions are the same as before:

$$\boldsymbol{F}_0(z)\boldsymbol{H}_0(z) + \boldsymbol{F}_1(z)\boldsymbol{H}_1(z) = 2z^{-\ell}\boldsymbol{I} \qquad (7.92)$$

$$\boldsymbol{F}_0(z)\boldsymbol{H}_0(-z) + \boldsymbol{F}_1(z)\boldsymbol{H}_1(-z) = \boldsymbol{0}. \qquad (7.93)$$

The big difference is that the anti-aliasing equation (7.93) is no longer satisfied by $\boldsymbol{F}_0(z) = \boldsymbol{H}_1(-z)$ and $\boldsymbol{F}_1(z) = -\boldsymbol{H}_0(-z)$. *The matrices $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ need not commute!* Probably they don't. A satisfactory construction method for PR (= biorthogonal) multifilters is not yet available.

For orthogonality the situation is similar. We want the synthesis filters to be the transposes $\boldsymbol{F}_0(z) = \boldsymbol{H}_0^T(z^{-1})$ and $\boldsymbol{F}_1(z) = \boldsymbol{H}_1^T(z^{-1})$ of the analysis filters (times a delay to make them causal). Omitting that delay, the PR conditions (7.92) require the modulation matrix to be paraunitary: $\boldsymbol{H}_m(z)\boldsymbol{H}_m^T(z^{-1}) = 2\boldsymbol{I}$. These matrices have order $2r$. The first block yields the famous Condition **O**:

$$\boldsymbol{H}_0(z)\boldsymbol{H}_0^T(z^{-1}) + \boldsymbol{H}_0(-z)\boldsymbol{H}_0^T(-z^{-1}) = 2\boldsymbol{I}. \qquad (7.94)$$

Suppose this is achieved — it is the hard part. Then in the scalar case, H_1 is the alternating flip of H_0. That no longer works. The lowpass and highpass rows

$$\begin{bmatrix} h(N) & h(N-1) & \cdots & h(1) & h(0) \\ -h(0) & h(1) & \cdots & -h(N-1) & h(N) \end{bmatrix}$$

are not orthogonal, if the $r \times r$ matrices $h(k)$ do not commute. We need to compute, from scratch, highpass coefficients $h_1(k)$ that will complete the paraunitary matrix $H_m/\sqrt{2}$.

This completion is possible [StSt1]. If we have r paraunitary rows of length $2r$, the factorization in equation (9.49) still exists. The constant matrix Q in that equation is $r \times 2r$. Complete it to a square constant orthogonal matrix. Then the factors multiply to give a square paraunitary matrix. Its last r rows contain the highpass coefficients we need.

Finally, we mention the accuracy p. As always, combinations of $\phi_i(t-k)$ must produce the polynomials $1, t, \ldots, t^{p-1}$. In the scalar case, $H(z)$ will have a factor $\left(1+z^{-1}\right)^p$. The matrix factorization is not so simple [CoDaPl]. Also in the scalar case, $M = (\downarrow 2)2H$ has eigenvalues $1, \frac{1}{2}, \ldots, \left(\frac{1}{2}\right)^{p-1}$. This is still the correct Condition A when the entries of M are $r \times r$ matrices:

$$M = 2\begin{bmatrix} h(N) & h(N-1) & \cdots & \cdots & & \\ & & h(N) & h(N-1) & \cdots & \cdots \end{bmatrix}.$$

The column sums $2\sum h(2k)$ and $2\sum h(2k+1)$ were 1 in the scalar case. They need not be I in the matrix case. That would be far too restrictive. To achieve $p = 1$, the sums must have $\lambda = 1$ as an eigenvalue with the same left eigenvector. The condition for higher p is recursive [StSt2,HeStSt]. In some way the scalar case is understood more deeply, when the theory of multiwavelets forces us into the matrix case.

Problem Set 7.5

1. **(Haar's Hat)** Find wavelets $w_1(t)$ and $w_2(t)$ on $[0, 1]$ that are orthogonal to each other and to $\phi_1(t) = \text{box}$ and $\phi_2(t) = 2t - 1$. They will be linear on $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$. Draw graphs of $w_1(t)$ and $w_2(t)$.

2. Express the wavelets in Problem 1 as combinations of $\phi_1(2t), \phi_1(2t-1), \phi_2(2t), \phi_2(2t-1)$. What are the highpass matrix coefficients $h_1(0)$ and $h_1(1)$?

3. Display a 4×4 block of the analysis bank and verify orthogonality:

$$\text{block} = \begin{bmatrix} h(0) & h(1) \\ h_1(0) & h_1(1) \end{bmatrix} = \begin{bmatrix} \text{lowpass} \\ \text{highpass} \end{bmatrix}.$$

4. Find the matrix dilation equation for the C^1 cubics $\phi_1(t)$ and $\phi_2(t)$ drawn in Figure 7.8. They are combinations of C^1 cubics on half-intervals.

5. The Fourier transform gives what product formula to solve the matrix dilation equation $\phi(t) = \sum 2h(k)\phi(2t-k)$?